Goal: Prove the excision axiom, which is equivalent to

**Theorem 2.2.2** Given subspaces $A, B \subset X$ with $X = \text{int}(A) \cup \text{int}(B)$, the inclusion $(B, A \cap B) \to (X, A)$ induces an isomorphism.

This follows from

**Prop 2.2.1** Given an open cover $U = \{U_1, U_2, U_3\}$, the...
$\mathcal{C}^U(x) \rightarrow \mathcal{C}(x)$

is a chain homotopy equivalence.

Key technical tool is barycentric subdivision.

There is a subdivision operation

$\mathcal{C}(x) \xrightarrow{\text{identification}} \mathcal{C}(x) \xrightarrow{\text{sum of the simplices}} \mathcal{C}(x)$

It is a CHF (see pages 121-123 of [Hatcher])
This means there is a chain homotopy \( D \) between \( S^1 \) and \( I_c(x) \). Let \( D_m \) be the chain homotopy for \( S^m \).

To prove 2.21 we need a map 
\[
C(x) \longrightarrow C(U(x))
\]

Given a simplex \( \sigma : \Delta^n \longrightarrow X \)

The covering of \( X \) pulls back to one of \( \Delta^n \), which can be replaced by a finite cover since \( \Delta^n \) is compact.
Introducing $m(\delta)$ times we get small spheres.

This means $S^m(\delta) \in \mathcal{C}^W(x)$. We define a new chain homotopy by $D(\delta) = D_m(\delta)$. We have a chain map $C(x) \xrightarrow{F} \mathcal{C}^W(x)$.
We need to show the isomorphisms

(1) \( C^U(X) \to C(X) \to C_U(X) \)

(2) \( C(X) \to C^U(X) \to C(X) \)

are each chain homotopic to \( I \).

For (1), \( \rho_L = I_{C^U(X)} \). For (2)

\( o \to s^{m(\ell)}(\ell) \)

\( D(\ell) = D_{m(\ell)}(\ell) \)
Can show $D$ is the desired chain homotopy. This proves $\widetilde{Z}_n \rightarrow Z_n$.

**New Topics:** 3 questions

1. Can we describe $\pi_\ast (X \times Y)$ in terms of $\pi_\ast (X)$ and $\pi_\ast (Y)$? Recall $\pi_\ast (X \times Y) = \pi_\ast (X) \oplus \pi_\ast (Y)$.

Example: $X = Y = S^1$

$X \times Y = \text{torus} = S^1 \times S^1 = T^2$

$\pi_\ast (S^1) = \{2 \text{ if } i = 0, 1 \}$

$\begin{cases} 0 & \text{if } i > 1 \end{cases}$
\[ H_i(T^2) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{if } i > 2 \end{cases} \]

Q: Let \( R \) be a commutative ring, replace \( C(X) \) by the corresponding free \( R \)-module \( CR(X) \).

\( H_x(X; R) \) (homology with coefficients in \( R \)) to be \( H_x(CR(X)) \). E.g. \( R = \mathbb{Q} \) or \( R = \mathbb{Z}/p \).

How does it relate to \( H_x(X) \)?
(3) Let \( A \) be an abelian group. Consider the homomorphism \( \text{Hom}(C_m(x), A) \Rightarrow C^n(X, A) \).

\[
\begin{align*}
C_{m-1}(x) & \xrightarrow{d_m} C_m(x) & \xrightarrow{d_{m+1}} C_{m+1}(x) \\
C^n(X, A) & \xrightarrow{\kappa} C^n(X, A) & \xrightarrow{\kappa} C^{n+1}(X, A)
\end{align*}
\]

Cohomology with coefficients in \( A \):

\[
H^*(X, A) = H^*(C^*(X, A))
\]
How does it relate to $H_c(X)$?

Homological algebra

All proofs are easy.