Let $A$ be an abelian gp and $C$ a chain complex of free abelian gts.

**Thm 3.1.** There is a natural short exact sequence

\[ 0 \to \text{Tor}_1(C, A) \to \text{Tor}_1(H_n(C), A) \to 0 \]

**Thm 3.2.** Ditto

\[ 0 \to \text{Hom}(H_n(C), A) \to H^1(\text{Hom}(C, A)) \to \text{Ext}^1(H_n(C), A) \to 0 \]

**Note**

i) $\text{Tor}_1 \neq 0$ only if $H_{n-1}C$ and $A$ have torsion.

ii) $\text{Ext}^1 \neq 0$ only if $H_nC$ has torsion.
Proof of 3. \( A_n \Rightarrow \)

\[
\begin{array}{cccccc}
0 & \rightarrow & Z_n & \rightarrow & C_n & \xrightarrow{dn} & B_{n-1} & \rightarrow & D \\
0 & \rightarrow & Z_{n-1} & \rightarrow & C_{n-1} & \xrightarrow{dm} & B_{n-2} & \rightarrow & 0
\end{array}
\]

where each row is exact, i.e. \( B_{n-1} \) is a direct summand of \( C_n \) since it is free abelian and hence projective.

We get a SES of chain complexes

\[
\begin{array}{cccccc}
0 & \rightarrow & Z & \rightarrow & C & \rightarrow & B & \rightarrow & D
\end{array}
\]

where \( Z \) and \( B \) have trivial boundary operators.

The LES of \( H_n \) is

\( \text{(1)} \)
\[ \cdots \to \mathbb{Z}_n \to \mathbb{H}_n C \to B_{n-1} \to \cdots \]

Tensoring 1 with \( A \) preserves exactness:
\[ 0 \to \mathbb{Z} \otimes A \to C \otimes A \to B \otimes A \to D \]

Also note that:
\[ 0 \to B_n \xrightarrow{\text{in}} \mathbb{Z}_n \to \mathbb{H}_n C \to 0 \]

is a free resolution of \( \mathbb{H}_n C \). We get a 6-term exact sequence of Tor groups:
\[ 0 \to \text{Tor}_1 (B_n, A) \to \text{Tor}_1 (\mathbb{Z}_n, A) \to \text{Tor}_1 (\mathbb{H}_n C, A) \]

\[ \xrightarrow{B_n \otimes A \xrightarrow{\text{in} \otimes A}} \mathbb{Z}_n \otimes A \to \mathbb{H}_n C \otimes A \to 0 \]
The LES in $H^*_B$ from $\mathbb{Z}$ is:

\[ \cdots \to Z_m \otimes A \to H_n(C \otimes A) \to B_{n-1} \otimes A \to \cdots \]

\[ \to Z_m \otimes A \to H_{n-1}(C \otimes A) \to B_{n-2} \otimes A \to \cdots \]

Hence we have a SES:

\[ 0 \to \text{coker } (i_m \otimes A) \to H_n(C \otimes A) \to \ker (i_{m-1} \otimes A) \to 0 \]

\[ \text{Tor}_1^{B_{n-1}}(H_{n-1}(C), A) \]

**QED.**

Proof of Theorem 3.12 is similar.

Example
This is related to \( \text{SO}(3) \).

\[ H^i \text{RP}^3 = \text{Hom}(\mathbb{Z}, \mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } i \neq 0 \text{ or } 2, \\ 0 \text{ if } i = 0 \text{ or } 2. \end{cases} \]

Apply \( \text{Hom}(\mathbb{Z}, \mathbb{Z}) \):

\[ \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \]

\[ H^i \text{RP}^3 = H^i \text{Hom}(\mathbb{Z}, \mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } i \neq 0 \text{ or } 2, \\ 0 \text{ if } i = 0 \text{ or } 2. \end{cases} \]
Theorem 3.2 says the Universal Coefficient Theorem:

\[ 0 \to \text{Hom}(H_2, \mathbb{Z}) \to H^2(\text{Hom}(C_2, \mathbb{Z})) \to \text{Ext}^1(H_1, C_2, \mathbb{Z}) \to 0 \]

\[ \therefore \]

\[ H_2(C_2, 2) = H_2(\mathbb{RP}^3, 2) = \begin{cases} 2/2 & \text{if } n \equiv 0 \pmod{2} \\ 2/2 & \text{if } n \equiv 1 \pmod{2} \end{cases} \]

\[ \text{UCT} \]
\(0 \to \mathbb{H}_2(C \otimes \mathcal{O} \to \mathcal{H}_2(C \otimes \mathcal{O} \to \mathcal{O}) \to \mathcal{T}_{\mathcal{O} \otimes \mathcal{A} \otimes \mathcal{O} \to \mathcal{O} \otimes \mathcal{A} \otimes \mathcal{O}} \to 0\)

Recall we want to describe \(\mathbb{H}_x (X \times Y)\) in terms of \(\mathbb{H}_x (X)\) and \(\mathbb{H}_x (Y)\). From there is natural SES.
Note: If either $H_n(X)$ or $H_n(Y)$ is torsion free, then $\text{Tor}_n$ vanishes.

Related algebraic statement:

Let $C$ and $C'$ be chain complexes of free abelian groups. There is a SES

\[ 0 \to H_i(c) \otimes H_{n-i}(c') \to H_n(c \otimes c') \to \text{Tor}_n(H_i(c), H_{n-i}(c')) \to 0 \]

The proof of this is similar to that of ZA. B. K"unneth formula.
Tricky part:

Let \( C = s(x) \) = angular chain \( x \) of \( E \)

\( C' = s(y) \) =

We know \( H_x (s(x) \otimes s(y)) \), but

\( s(x) \otimes s(y) \neq s(x \times y) \)