$\mathbb{C}W$ complex = cellular chain complex.

$\mathbb{R}P^n = \{ [x_0, x_1, \ldots, x_n] : x_i \in \mathbb{R} \}$, where

$[\lambda x_0, \ldots, \lambda x_n] = [x_0, \ldots, x_n]$ for $\lambda \neq 0$,

$(x_0, \ldots, x_n) \neq (0, \ldots, 0)$.

We get a $CW$ complex with one cell in each dimension from 0 to $n$ and a chain complex $C$ with

$C_i = \mathbb{Z}^2$ for $0 \leq i \leq n$,

$C_i = 0$ for $i > n$.

To compute $\partial_1$.
\[ S^{i-1} \xrightarrow{\text{attaching map}} \mathbb{R}P^{i-1} \to \mathbb{R}P^{i-1}/\mathbb{R}P^{i-2} = S^{i-1} \]

**Claim** \[ \deg g_i = 1 + (-1)^i \]

**Covering** \[ \deg d_i = \begin{cases} 2 & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases} \]

This is related to the fact that the degree of the antipodal map on \( S^{n-1} \) is \( (-1)^n \)

\[ (-1)^n = \det \text{of } -I, \text{ where } I \text{ is the } n \times n \text{ identity matrix.} \]

Our chain complex:

\[ 2 \xleftarrow{0} 2 \xrightarrow{2} 2 \xleftarrow{0} 2 \xrightarrow{2} \cdots \]
\[ H_i(\mathbb{R}P^n) = \begin{cases} 
2 & \text{for } i = 0 \\
2^{1/2} & \text{for } i \text{ odd and } 0 < i < n \\
2 & \text{for } i = n \text{ and } n \text{ odd} \\
0 & \text{otherwise} 
\end{cases} \]

Cartesian products:

Let \( X \) and \( Y \) be CW complexes with cellular chain complexes \( C(X) \) and \( C(Y) \).

Then there is a CW-structure on \( X \times Y \) such that \( C(X \times Y) = C(X) \otimes C(Y) \).

To see this, \( X \) is union of the interiors of its cells.
The product of an \( i \)-cell in \( X \) and a \( j \)-cell in \( Y \) is an \((i+j)\)-cell in \( X \times Y \). Hence the Künneth theorem can be proved painlessly.

Cup products

\[
H^i(x) \otimes H^j(y) \longrightarrow H^{i+j}(x) 
\]

Recall we have map

\[
H_x(x) \otimes H_x(y) \longrightarrow H_x(x \times y) 
\]

It is part of a split exact sequence so we also have
\[ H_x(X) \otimes H_x(Y) \xrightarrow{\text{cap product}} H_x(X \times Y) \]

Let \( X = Y \) and \( \Delta : X \to X \times X \)
\[ \Delta(x) = (x, x) \]

\[ H^*(X) \otimes H^*(X) \to H^*(X \times X) \xrightarrow{\text{cap product}} H^*(X) \]

Formal properties
1) Natural in \( X \)
2) Cococentric + distributive
3) There is unit in \( H^0 \), namely
4) It is commutative up to sign
For $\alpha \in H^i(X)$, $\beta \in H^j(X)$
$\beta \alpha = (-1)^{ij} \alpha \beta = \delta_{ij} \alpha \beta$ if $i = j$
$\beta \alpha = -\alpha \beta$ if $i$ and $j$ are odd

Examples

Let $x \in H^2(CP^n)$ be a generator
then $x^k$ generates $H^{2k}(CP^n)$ for $k \leq n$. 
As a graded ring,
\[ H^*(CP^n) = 2[x] / (x^n) \quad \text{where} \quad x \in H^2 \]

Similarly,
\[ H^* (RP^n; \mathbb{Z}/2) = \mathbb{Z}/2[y] / (y^{n+1}) \]

where \( y \in H^2 \)

Application:
Consider the attaching map for the 4-cell in \( CP^2 \). This is the Hopf map:
\[ S^3 \to S^2 = CP^1 \quad \text{in} \quad \pi_3 (S^2) \]

\[ (x, y) \mapsto [Ex, y] \quad \text{where} \quad x, y \in \mathbb{C} \]
Claim $\pi$ represents a nontrivial element in $\pi_3(S^2)$.

Suppose it is trivial. Then $\mathbb{C}P^2 \vee S^2 \vee S^4 = W$.

We have maps $S^2 \to W \to S^2$ and $S^4 \to W \to S^4$.

We have generators $x \in H^2(W)$ and $y \in H^4(W)$. $\pi$ implies $x^2 = 0 \in H^4(W)$.

Our map $W \to S^2$ induces an
\[ \text{Hence } x^2 = 0 \text{ in } H^4 W, \text{ but } x^2 \neq 0 \text{ in } H^4 CP^3, \]

\[ \eta \text{ is essential} \]

Hirzebruch proved \[ \pi_2(\mathbb{S}^2) \cong \mathbb{Z} \] he defined a map \[ \pi_2(\mathbb{S}^2) \to \mathbb{Z} \]

as follows:

\[ \mathbb{S}^3 \xrightarrow{f} \mathbb{S}^2 \]
Let $x$ be the CW-complex obtained from
\[ X = S^2 \cup e^4 \]
\[ H^i X = \mathbb{Z}^2 \quad \text{for } i = 0,2,3,4 \]
\[ H^1 X = 0 \quad \text{else} \]

Let $x$ and $y$ generated $H^2$ and $H^4$.
Then $x^2 = ny$ for some $n \in \mathbb{Z}$.

$n$ is the Hopf invariant $\tilde{H}(f)$ of $f$.
Hopf showed that this defines an isomorphism $\pi_3 S^2 \to \mathbb{Z}$.

Generalization:
$S^{4n-1} \rightarrow S^{2n}$

Then we get a hom $\Phi_{4n} : S^{2n} \rightarrow \mathbb{Z}$
derived from the cup product in $H^*X$ where $X = S^{2n} \cup C^{4n}$.

For $n > 1$ it has a non-trivial kernel.

Is it onto ??? Hofk showed it is for $n = 1, 2$ on $H$.

For $n = 2$ we can define a map similar to $N$ using quaternion.
$S^7 = \text{unit sphere in } H \cong \mathbb{R}^8$

$S^4 = \text{quaternionic projective line}$

$S^7 \xrightarrow{\nu} S^4 \quad \ni x, y \in H$

$(x, y) \mapsto [ix, y]$  

For $n = 4$ we have the octonions or Cayley numbers $\mathbb{O} \cong \mathbb{R}^8$

This leads to a map $S^1 \cong S^7 \to S^4$ and $6 \in \Pi_{15} S^8$

Each has Hopf invariant 1.