Recall the vector field problem.
Let \( n+1 = 2^k \) s odd.
\( S^n \) has \( \phi(k) \) vector fields
where \( \phi(k) = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k = 1 \\ 3 & \text{if } k = 2 \\ 7 & \text{if } k = 3 \\ 6 + \phi(k-4) & \text{if } k \geq 4 \end{cases} \).

\( S^n \) has \( 1 + \phi(k) \) \( \mapsto \) a certain map
\( \beta_k : S^n \rightarrow S^n \phi(k) \) is null homotopic.
Call this the Adams criterion.
To describe this map:
Def. The J-homomorphism \( \pi_n \text{SO}(n) \to \pi_{n+1} \mathbb{S}^n \) for \( k, n > 0 \) is defined as follows. Given \( s^k \to \text{SO}(n) \), define a map

\[
S^k \times D^n \to D^n
\]

\[
(x, y) \mapsto f(x)(y)
\]

\[
S^{n+k} = 2 \cdot (D^{n+k+1}) = 2 \cdot (D^{k+1} \times D^n)
\]

\[
= 2D^{k+1} \times D^n \cup D^{k+1} \times 2D^n
\]

\[
= S^k \times D^n \cup D^{k+1} \times S^{n-1}
\]

\[
\text{e.g., } n = k = 1 \quad S^2 = (S^1 \times D^1) \cup D^2 \times S^0
\]
$S^k \times D^n \rightarrow D^m$

$S^{n+k} \xrightarrow{J(g)} D^m / \partial D^m = S^n$

$J(g)$ is the extension of $g$, obtained by sending the rest of $S^{n+k}$ to the basepoint in $S^n$. 

Remark: $\pi_k SO(n) \xrightarrow{J} \pi_{n+k} S^n$. Both groups are independent of $n$ if
\( n = k + 1 \). Denote the gap on the left for each \( n \) by \( T_k \).

**Theorem (Bott 1959)**

\[
T_k \equiv 0 \mod 8 \\
T_k \equiv 2 \mod 2 \\
T_k \equiv 0 \mod 1
\]

**Periodicity**

- \( k = 0, 1 \) mod 8
- \( k = 3, 7 \) otherwise

**Theorem (Adams + others)**

\( J \) maps each nontrivial gap above onto a summand of \( T \) in \( S^n \). The order of \( J \) is 50 in a \# \# \#, for which there is a number theoretic formula related to the value of the Riemann...
Beta function at negative odd integers

\[ \frac{1}{2 \cdot 2 \cdot 3} \text{ etc.} \]

\[ a_1, a_2, a_4, \ldots \]

What is \( \beta_{n-\phi(k)} \) ?

\[ \beta_{n-\phi(k)} = \prod_{k \in \mathbb{N}} s^{n-\phi(k)} \]

If \( n \equiv 1, 3, 7, 11, 15 \mod 15 \), then these data depend only on \( \phi(k) \), i.e., \( n - \phi(k) \equiv 1 + \phi(k) \mod 15 \)

Both ways \( \pi(k) = 0 \) if \( \phi(k) \)

\[ = 0, 1, 3, 7 \mod 15 \] in each case
The Adams map \( \beta_k \) is the image under \( J \) of the generator of \( \tilde{H}_0(\mathbb{R}) \) \( SO_3 \), i.e., it is nontrivial.

**END OF VECTOR FIELD**

**NEXT TOPIC**: Spectral sequence. Suppose we have a fiber bundle \( F \to E \to B \).

- \( F = \tilde{p}^{-1}(b) \) for \( b \in B \). How to describe \( H_*E \), \( E \) in terms of \( H_*F \) and \( H_*B \)??

Example \( E = F \times B \). We know the
In particular, \( H^*(E; k) = H^*(B; k) \otimes_k H^*(F; k) \)
for a field \( k \), e.g., \( k = \mathbb{Q} \) or \( \mathbb{Z}/p \).

Starting point of Serre spectral sequence is a bigraded group

\[ E_2^{st} = H_0(B; H^*_F) \]

on \( H_0(B; H^*_F) = H_0(B; k) \otimes_k H^*_F(k) \)

on similar ops defined in terms of \( H^*_B \) and \( H^*_F \). There is a mechanism that will lead us to \( H^*_X(x) \) on
$H^*(X)$ possibly with coefficients in $k$.

Magic:

There is a homomorphism $E_2^{s,t} \xrightarrow{d_2} E_2^{s-2, t+1}$ such that $d_2 \circ d_2 = 0$ when defined. We get a chain complex and we want its homology:

$E_2^{s,t} = \ker d_2 / \text{im } d_2^{s+2, t-1}$

There are homomorphisms $E_3^{s,t} \xrightarrow{d_3} E_3^{s-2, t+2}$

$d_3 \circ d_3 = 0 \quad \text{etc.}$
\[ E_{m} \stackrel{d_{n}}{\rightarrow} E_{n}^{m-n} \]

with \( d_{m} \circ d_{n} = 0 \) and \( E_{m+1}^{0,0} = \ker d_{m} / \im d_{m} \)