Recall that for a closed \( n \)-manifold \( M \) we have \( Sw \ w^\ast (M) \in H^n (M; \mathbb{Z}) \).

Let \( x \) be a product of them in \( H^n (M) \).

Then \( \langle x, [M] \rangle \in \mathbb{Z}/2 \) is a SW number of \( M \). All such \#s vanish if \( M \) is an boundary.

Variations: Let \( M \) be oriented. Then there is a fundamental class \([M] \in H^n (M; \mathbb{Z})\).

For \( 4/n \) then we can defined Poincaré \#s in the same way.

These vanish if \( M \) is an oriented boundary.
Suppose \( \nu(m) \) has a complex structure. (This implies orientability and \( n \) even.) Then we can define Chern \#s. I can replace \( \nu(m) \) by a normal bundle \( \nu' \). For \( \mathbb{R}^m \hookrightarrow \mathbb{R}^{m+2k} \) and \( n^m = 2(n^{m+1} - 1) \). \( \nu^m \hookrightarrow \mathbb{R}^{2m+k} \times I \).)

We get Chern \#s.

[Thom, Milnor, Novikov]
Def. Two $n$-manifolds $M_1, M_2$ are cobordant if there is an $(n+1)$-mfd $N$ with $\partial N = M_1 \sqcup M_2$. In oriented case $\partial N^{n+1} = M_1 \sqcup \overline{M}_2$ where $\overline{M}_2$ is $M_2$ with its orientation reversed.

Cobordism (as defined above) is an equivalence relation.
The set of equivalence classes is a graded ring under disjoint union (addition) and Cartesian product (multiplication).

In the unoriented case, it is also a 2/12 vector space. Given any closed manifold $M$, $\omega(I \times M) = M \cup M$

Oriented case

$\omega(I \times M) = M \uplus \overline{M}$
We get 3 cobordism rings

0. Unoriented case (studied by Thom in 1950).

\[ \Omega_k = \pi_k = \mathbb{Z}/2 \mathbb{Z} \left[ x_2, x_4, x_6, x_8, \ldots \right] \]

where \( \dim x_i = i \) and \( i \neq 2^k - 1 \).

Every closed 3-manifold is a boundary.

Remark. The even dimensional generators can be the \( \mathbb{R}P^{2n} \). Odd dimensional generators are harder to describe.
1. Oriented case. $\mathcal{S}^{50}_x \otimes \mathcal{M} 50_x$ is awkward to describe. It includes some 2-torsion and a torsion quotient:

$$\mathcal{S}^{50}_x \otimes \mathcal{Q} = \mathcal{Q} \left[ \gamma_1, \gamma_2, \gamma_3, \ldots \right]$$

where $\dim \gamma_i = 4i$

$$\gamma_i = \left[ \mathbb{C} P^{2i} \right]$$

2. Complex case.

$$\mathcal{M}V_x = \mathcal{S}^{10}_x = \mathbb{Z} \left[ z_1, z_2, \ldots \right]$$

where $|z_i| = 2i$

$$\mathcal{M}V_x \otimes \mathcal{Q} = \mathcal{Q} \left[ \mathbb{C} P^1, \mathbb{C} P^2, \mathbb{C} P^3, \ldots \right].$$
How to prove the theorem.

Let \( E \) be a \( D^n \)-bundle over \( X \).
[We get such a thing from any \( R^n \)-bundle over \( X \).]
If \( X \) is a mfd, then \( DE \) is an \( S^{n-1} \)-bundle over \( X \).

The Thom space \( T \) is the quotient \( E/DE \).

Alternate description.

Let \( E \) be an \( R^n \)-bundle over a compact space \( X \). Let \( T \) be its 1-point compactification.
Thom isomorphism: Thm

\[ H^i(X; \mathbb{Z}/2) \cong H^{n+1}(T; \mathbb{Z}/2) \]

and \( T \) is \((n-1)\)-connected.

Remark. \( H^n(E, \mathbb{Z}/2E) \cong \text{Thom class} \xrightarrow{\ast} X \)

\[ D^n \to E \]

\[ S^{n-1} \to \Omega E \]

If the bundle is oriented (meaning the structure group is \( \text{SO}(n) \)), there is class \( u \in H^n(E, \mathbb{Z}/2E) = H^n(T) \) that
make to a generator of \( Z \). If bundles is not oriented this only works mod 2.

Given \( x \in H^i(X) = H^i(E) \),

\[ u x \in H^{n+i}(E, \mathcal{A}) = H^{n+i}(\mathcal{A}) \]

The cup product gives the Thom isomorphism.