What happens to the rotating coffee cup?

It follows a closed path in $\mathbb{R}^3$ and in $SO(3)$, the group of $3 \times 3$ orthogonal matrices with determinant $1$.

We have a closed path in $\mathbb{R}^3 \times SO(3)$, and $\mathbb{R}^3$ has a standard topology. $SO(3)$ is a subset of the set of all $3 \times 3$ matrices in $\mathbb{R}^9$. $SO(3)$ inherits a topology from $\mathbb{R}^9$. 
The motion of the cup is described by a continuous map \( I = [0, 1] \to X \) with \( f(0) = f(1) \).

**Definition** Two maps \( f_0, f_1 : X \to Y \) are homotopic if they can be continuously deformed to \( f_1 \), i.e., there is a map \( h : I \times X \to Y \) such that \( h(0, x) = f_0(x) \) and \( h(1, x) = f_1(x) \).

\( h \) is called a homotopy between \( f_0 \) and \( f_1 \), and we denote \( f_0 \simeq f_1 \).
**Examples**

\[ Y = S^1 = \text{circle} \]

\[ \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \} \]

\[ Y = \mathbb{R}^2 \setminus \{ (0, 0) \} \]

Images of \( f_1, f_2 \) are \( f_3 \)

Easy lemma: \( \forall A \subseteq Y = \mathbb{R}^n \), then any \( 2 \) maps \( f_0, f_1 : X \to Y \) are homotopic.
Proof: Let \( h(t, x) = (1-t) f_0(x) + t f_1(x) \),
where \( 0 \leq t \leq 1 \). Then
\[
h(0, x) = f_0(x) \text{ and } h(1, x) = f_1(x).
\]
QED.

Note: The coffee cup map
\[
f: I \rightarrow X = \mathbb{R}^3 \times SO(3) \rightarrow SO(3)
\]
is homotopic to the constant map.

What can we say about closed paths in \( SO(3) \)?

The left trick also illustrates
a homotopy between 2 closed paths in \( \text{SO}(3) \).

Consider \( \text{SO}(2) \xrightarrow{\theta} \text{SO}(3) \).

\[
\theta \\
\left[
\begin{array}{cc}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta \\
\end{array}
\right] \xrightarrow{\theta} \\
\left[
\begin{array}{ccc}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{array}
\right]
\]

If we let \( \theta \) range from 0 to \( 2\pi \) we get a closed path in \( \text{SO}(2) \) and in \( \text{SO}(3) \).

\( \text{SO}(2) \times S^1 \)
Will see later that this closed path in $SO(2)$ is not homotopic to the constant map. We say it is essential. (opposite: null homotopic)

Variation: Let $0 \leq \theta \leq 2\pi n$ for a positive integer $n$.

Facts to be proved later:
1. This path in $SO(2)$ is essential for all $n > 0$.
2. The closed path in $SO(3)$ is null homotopic $\iff n$ is even.