Recall \( \pi_1(X, x_0) = \) homotopy classes of maps \((I, \partial I) \to (X, x_0)\) equivalence \((s^1, x_0) \to \)

\[\tilde{T}_n(X, x_0) = \text{h.top. classes of maps} \]

\[\left( I^n, \partial I^n \right) \to (X, x_0)\]

Set \((t_1, t_2, \ldots, t_n) \in I^n \text{ with } 0 \leq t_i \leq 1 \text{ given 2 such maps } f \text{ and } g \)
Define $f \times g : (t_1, \ldots, t_n) \mapsto f(2t_1, t_2, \ldots, t_n)$ if $0 \leq t_1 \leq \frac{1}{2}$, or
$g(2t_1, t_2, \ldots, t_n)$ if $\frac{1}{2} \leq t_1 \leq 1$.

This leads to a group structure as before so we have $\pi_n(X, x_0)$.

Then $\pi_n(X)$ is abelian for $n=1$, i.e. $f \times g = g \times f$.

Proof: $f \times g, g \times f : (I^n \times I^n) \to (X, x_0)$ is a homotopy between them as a map $h : I \times (I^n \times I^n) \to (X, x_0)$
$(s, t_1, t_2, \ldots, t_n)$ for $0 \leq s \leq 1$.
$d = 3/4$

$D = \frac{7}{3}$

$A = 1$

**General remarks:**

$\Pi_m(x)$ is easy to define but hard to compute.
\[ \prod_i (S^n) = \{ \} \quad \text{so for } i < n \]
\[ \prod_i (S^n) = \{ \} \quad \text{for } i = n \]
\[ \prod_i (S^n) \neq 0 \quad \text{for } i \geq n \]

Will describe some methods for finding \( \prod_i \).

Later we will define \( H_m(X) \).
(homology), harder to define but easier to compute.

A map \( (x, x_0) \to (y, y_0) \) induces homomorphisms \( \pi_n(x, x_0) \to \pi_n(y, y_0) \).

If \( f \) and \( f' \) are two such maps that are homotopic, then \( \pi_n(f) = \pi_n(f') \).

Read Chapter 0 and hand in homework Friday.

Def: A map \( X \to Y \) is a homotopy
equivalence if there is a map \( Y \to X \) such that \( \exists f \), \( \exists g \) such that \( X \xrightarrow{g} Y \xrightarrow{f} X \), \( f \circ g \), \( g \circ f \), \( \text{identity on } X \), and \( f \circ g = 1_Y \). We say \( X \) and \( Y \) have the same homotopy type.

Example: \( X = \text{pt} \), \( Y = \mathbb{R}^n \), \( f = \text{inclusion of origin} \), \( \text{pt} \xrightarrow{f} \mathbb{R}^n \xrightarrow{0} \text{pt} \xrightarrow{f} \mathbb{R}^n \).
Then \( gh = 1_X \) but \( fg \neq 1_Y \).

\[
h: I \times \mathbb{R}^n \to \mathbb{R}^n \quad 0 \leq t \leq 1
\]

\[
(t, x) \mapsto t \cdot x
\]

\( h \) is the desired homotopy. \( \mathbb{R}^n \) is contractible.

**Def**: \( X \) is contractible if it is homotopic to a point.

**Def**: A subspace \( A \subseteq X \) is a retract of \( X \) if there is a map \( X \to A \) s.t.

\[
A \xrightarrow{i} X \xrightarrow{r} A \text{ is homotopic to } 1_A.
\]
Example  \( X = A \times B \)  

\[ A \overset{i}{\to} X = A \times B \overset{p_1}{\to} A \]
\[ a \mapsto (a, z) \]
\[ (a, b) \mapsto a \]

Example  \( A = S^1 \), \( X = \) Mobius strip

\[ \pi_1(A) = \mathbb{Z} \]
\[ \pi_1(X) = \mathbb{Z} \]
\[ \pi_1(B) = \mathbb{Z} \]

The map  \( B \to X \to A \)  
\[ \mathbb{Z} \overset{2}{\to} \mathbb{Z} \overset{1}{\to} \mathbb{Z} \]

induced in  \( \pi_1 \)