To finish proof of von Kampen Theorem
Let $x \in X \implies x = x' A U B$ with $x' \in A \cap B$. $A$, $B$ and $X$ are each path connected. Then $\pi_1 X$ is the pushout $P$ of the diagram

\[
\begin{array}{c}
\pi_1(A) \xrightarrow{s'} \pi_1(A \cap B) \\
\pi_1(A \cap B) \xrightarrow{s} \pi_1(B)
\end{array}
\]

and $\pi_1 X$ is into. Need to show it is 1-1.
Let $f$ and $f'$ be closed paths in $X$ with factorizations

$$f \sim f_1 \ast \cdots \ast f_m$$

and

$$f' \sim f'_1 \ast \cdots \ast f'_n$$

where each $f_i$ and $f'_j$ are closed paths in $A$ or $B$. Assume $f \sim f'$. Next to show their factorizations are equivalent.

Let $h : I^2 \to X$ be a homotopy between $f$ and $f'$

$$h(0,0) = f(a)$$

$$h(1,0) = f'(a)$$

$$h(0,1) = f'(b)$$

$$h(1,1) = f(b)$$

$$h(0,b) = h(1,b) = x_0$$
We can find $0 = x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_{k-1} \leq x_k = 1$

and $0 = t_0 \leq t_1 \leq \ldots \leq t_{k-1} \leq t_k = 1$

such that $R_{ij} = [x_{i-1}, x_i] \times [t_{j-1}, t_j]$ gets sent to $A$ or $B$. \[ f^0 = 6 \]

$a =$ path parameter
$t =$ homotopy parameter
$x =$
Let $W_{ij} = \{ x_i, y_j \} \in J^2$ be any point call $R_{i,j}$ $R_{i,k}(j-1)$. Each $w$ is surrounded by $\leq 4$ rectangles. Can choose a path $g_{i,j}$ from $h(W_{i,j})$ to $x_0$ s.t. if the surrounding rectangles do not all map to the same subspace then $h(W_{i,j}) \in A \cap B$ and we choose the path $g_{i,j}$ to lie in $A \cap B$.

Let $f_0 = f$ and $b_{p+1} = f'$. For $0 \leq p < k$ define $b_p$ to be the path given by
restricting  \( h \) to the path in  \( I \) that separates the first \( p \) rectangles from the others. Then we have
\[
f = f_0 \circ f_1 \circ f_2 \cdots \circ f_{k-1} \circ f_k = f'.
\]
Each \( f_p \) has a factorization (using the paths \( g_{i,j} \) as before).
The factorization of \( f_{p_j} \) is equivalent to the one for \( f_p \) in the sense of \( \|21\|_3 \).
In each case we are replacing one
product in $\pi_1 A$ or $\pi_1 B$ by another. This implies that the homotopic paths $f$ and $f'$ have equivalent factorizations. This means $E$ is 1-1.

QED

More examples of VKT in action

$X =$ surface of genus 2

$A, B$ as shown

$A \cap B = \emptyset$
\[ A = \begin{align*} & \leq S' \cup S^1 \leq \\ \mapsto & \Pi_1 A = F_2 = \langle a_1, a_2 \rangle \\
 & \Pi_1 B = F_2 = \langle b_1, b_2 \rangle \end{align*} \]

\[ \mathcal{N} = \Pi_1 (A \wedge B) \]

\[ \langle x \rangle \]

\[ t_1, b_2, b_1, b_1^{-1} b_2^{-1} = [b_1, b_2] \]

It follows that \( \Pi_1 X = \langle a_1, a_2, b_1, b_2; [a_1, a_2], [b_1, b_2] \rangle \) is non-abelian.
For a surface of genus $g$.

$X_4 = \langle a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \rangle$

$[a_1, b_1], [a_2, b_2], [a_3, b_3], [a_4, b_4] = e$