Recall chain complexes.

Question: Given two chain maps \( f, g : C \rightarrow C' \), when do they induce the same map in homology?

Later for \( X \xrightarrow{f} Y \) for spaces \( X, Y \), we will have \( C(X) \xrightarrow{c(f)} C(Y) \). The homotopy axiom says that if \( f \simeq g \) then \( H_n(f) = H_n(g) \).
Def. Two chain maps $C \xrightarrow{f} C'$ are chain homotopic if there is a collection of maps $D_n : C_n \rightarrow C_{n+1}'$ such that

$$d'_{n+1}D_n + D_{n-1}d_n = f_n - g_n$$

Then we say $\{D_n\}$ is a chain homotopy between $f$ and $g$. 
Then if $f$ and $g$ are chain homotopic,

then $H_n(f) = H_n(g)$

**Proof** Let $x$ be a cycle $c_0 C$ be represented by a cycle $x \in C_n$. Then

$(f_n - g_n)(x) = (d_{n+1} D_n + D_{n-1} d_n) x$

$= d_{n+1} D_n x$ since $d_n x = 0$

$= 0$ boundary in $C_n$.  
\[ \beta_n(x) - \gamma_n(x) = 0 \quad \text{in} \quad H_x C \]

\[ \beta_n(x) = \gamma_n(x) \in H_n C \quad \text{QED} \]

Towards topological homology

**Def.** The standard \( n \)-simplex

\[ \Delta^n = \{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0 \quad \sum x_i = 1 \} \]
E.g., $n=1$

$n=2$

$\Delta^2 = \text{area}\triangle$
$n = 3$  \hspace{1cm} \Delta^3 = \text{Tetrahedron}$

Alternate definition

$\Delta^n = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq 1 \right\}$

$n = 2$

$\mathcal{S}(x,y) : 0 \leq x \leq y \leq 1$
Def: The $j$th face of $\Delta^n \subset \mathbb{R}^{n+1}$ as above is the set of pts with $x_j = 0$, where $0 \leq j \leq n$.

Denote this by $\Delta^n_j \cong \Delta^{n-1}$.

Def: A $\Delta$-complex $\Sigma$ is a union of simplices of random dimensions by gluing together.
along fact

Example

terms $T^2$

IRP$^2$

Klein bottle

$\Omega$
In (f) we have one 0-complex \( \{ v_5 \} \).

The 1-complexes \( \{ a_1, b_5, c_3 \} \).

Two 2-complexes \( \{ U, L \} \).

Def. Given a \( \Delta \)-complex \( X \), we define a chain complex \( C(X) \) by \( C_n(X) \) as free abelian group on the set of \( n \)-simplices.
Given \( \Delta^n_j \mapsto \Delta^n_k \mapsto X \quad 0 \leq j \leq n \)

To define \( C_\infty^{(n)} \xrightarrow{d_n} C_{n-1}^{(n)}(X) \)

\[
[x] \mapsto \sum_{0 \leq j \leq n} \Delta^n_j[x] \in \Delta^n_k
\]

Lemma: For \( C_\infty^{(n)}(X) \) as above,

\( d_{n-1} \circ d_n = 0 \).
Proof: Exercise.

Computation for $\Omega$:

$C_2(X) = \mathbb{Z} \oplus \mathbb{Z}$ with gens $[U]$ and $[L]$

$C_1(X) = \mathbb{Z}^3$ on gens $[a], [b], [c]$

$C_0(X) = \mathbb{Z}$ on $[\nu]$

$d_1 [a] = 0 = d_1 [b] = d_1 [c]$
\[ d_2 [U] = [b] - [c] + [a] \]

The \( j \)th term is the edge obtained by removing the \( j \)th vertex.

\[ d_2 [L] = [a] - [c] + [b] \]

\( H_0 = C_0 = \mathbb{Z} \)

\( H_1 = \ker d_1 / \im d_2 \)
\[ H_2 = \frac{2^3}{3} = \frac{2^2}{3}, \]

\[ \text{key } A_2 = \text{subject} \quad \{uv\} = \{uv\} \]

= 2