Given an $R$-module $M$, let $P_\bullet \to M$ be a projective resolution (shortened as $0 \to M \to P_0 \to P_1 \to \cdots$).

Then $\Tor_*^R(M,N) = H_* \left( \mathcal{A} \otimes_R N \right)$

$$ \Ext_*^R(M,N) = H_* \left( \Hom_R(P_\bullet,N) \right) $$

for any $R$-module $N$.

Toward long exact sequence for $\Tor$ and $\Ext$

Let $0 \to M' \to M \to M'' \to 0$.

We want a short exact sequence of resolutions

$$ 0 \to P_\bullet' \to P_\bullet \to P_\bullet'' \to 0 $$
Let $P_0 = P_0' \otimes P_0''$. Easy to show the map $P_0 \to M$ is onto. Note that the map exists because $P_0''$ is proj.
Choose surjections $P'_1 \to M'_1$ and $P''_1 \to M''_1$ and define $P'_1 = P'_1 \oplus P''_1$. This leads to a projective resolution $P_\ast$ of $M$ with $P'_1 = P'_1 \oplus P''_1$. Hence we have

$$0 \to P'_1 \to P_\ast \to P''_1 \to 0$$

as desired. Then we have

$$P_i \otimes N = P'_i \otimes N \oplus P''_i \otimes N$$
and

$$\text{Hom} (P_i, N) = \text{Hom} (P'_i, N) \oplus \text{Hom} (P''_i, N)$$

$P_\ast$ need not be $P'_1 \oplus P''_1$ as a chain or even though $P'_1 = P'_1 \oplus P''_1$ as modules.
We have short exact sequences of chain complexes

\[ 0 \rightarrow \mathcal{P}_x \otimes N \rightarrow \mathcal{P}_x \otimes N \rightarrow \mathcal{P}_x'' \otimes N \rightarrow 0 \]

\[ 0 \leq \text{Hom}(\mathcal{P}_x', N) \leq \text{Hom}(\mathcal{P}_x, N) \leq \text{Hom}(\mathcal{P}_x'', N) \leq 0 \]

These lead to the derived long exact sequences in Tor and Ext.

**Question:** Given a chain complex $C$ and an abelian $A$, what is $H_x (C \otimes A)$ in terms of $H_x (C)$ and $A$? And what is $H^x (\text{Hom} (C, A))$?
Thm 3 A.3 Let $C$ be a chain complex of free abelian groups and $A$ an abelian group. Then there is a functorial short exact sequence

\[ 0 \to H_n(C) \otimes A \to H_n(C \otimes A) \to \text{Tor}_n(H_{n-1}(C), A) \to 0 \]

1. A map $C \to C'$ on $A \to A'$ induces a map of short exact sequences.
2. It is common to abbreviate $\text{Tor}_n(-, -) = \text{Tor}_n(-, 0)$. Then $\text{Tor}_n(H_{n-1}(C), A) \neq 0$ only if both $H_{n-1}(C)$ and $A$ have torsion.
3. $H_n(C \otimes A) = H_n(C) \otimes A \oplus \text{Tor}_n(H_{n-1}(C), A)$
But the splitting is not functional.

**Theorem 3.2** For $C$ and $A$ as above there is a functional short exact sequence:

$$0 \to \text{Hom}(C,A) \to H^n(C;A) \to \text{Ext}^n(H_{n-1}C;A) \to 0$$

These are called universal coefficient theorems.

**Proof of 3A.2.** We have a SES:

$$0 \to Z_n \to C_n \xrightarrow{\partial_n} B_{n-1} \to 0$$

$$0 \to Z_{n-1} \to C_{n-1} \to B_{n-2} \to 0$$
We can define chain $Z$ and $B$ where

$$(Z)_n = Z_n = \ker \partial_n$$

$$(B)_n = B_{n-1} = \text{im} \partial_n$$

We have a SES of chain complex

$$0 \to Z \to C \to B \to 0$$

$Z$ and $B$ have trivial boundary as

so $H_n(Z) = Z_n$ and $H_n(B) = B_{n-1}$.

From $(*)$ we get a LES in homology

$$\cdots \to H_{n+1}B \to H_nZ \to H_nC \to H_nB \to H_{n-1}Z \to \cdots$$

$0 \to B_n \to Z_n \to H_nC \to B_{n-1} \to \cdots$
Each row of (1) is split, i.e.

\[ C_n = \mathbb{Z}_n \oplus B_{n-1} \]

so

\[ C_n \otimes A = (\mathbb{Z}_n \otimes A) \oplus (B_{n-1} \otimes A) \]

The short exact sequence

\[ 0 \rightarrow B_n \xrightarrow{\text{in}} \mathbb{Z}_n \rightarrow H_n(C) \rightarrow 0 \]

is a projective resolution of \( H_n(C) \). Hence we get a 6-term sequence of Tor and Ext. Since \( B_n \) and \( \mathbb{Z}_n \) are free abelian,

\[ \text{Tor}_1 (B_n, A) = 0 \text{ and } \text{Tor}_1 (\mathbb{Z}_n, A) = 0. \]

We have a 4-term exact sequence

\[ 0 \rightarrow \text{Tor}_1 (H_n C, A) \rightarrow B_n \otimes A \rightarrow \mathbb{Z}_n \otimes A \rightarrow H_n(C) \otimes A \rightarrow 0 \]
Since $C_n = \mathbf{Z}_m \otimes R_{m-1}$, Tensoring with $A$ gives a SES of chain complexes

$0 \to \mathbf{Z} \otimes A \to C \otimes A \to B \otimes A \to 0$

Note

$H_n(\mathbf{Z} \otimes A) = \mathbf{Z}_m \otimes A$

$H_n(B \otimes A) = R_{m-1} \otimes A$

and $H_n(C \otimes A) = ???$

The LES in $H_n$ for $\otimes$ is

$\cdots \to H_n(\mathbf{Z} \otimes A) \to H_n(C \otimes A) \to H_n(B \otimes A) \to H_{n-1}(\mathbf{Z} \otimes A) \to \cdots$

$\text{ker}(\text{in} \otimes A) \to H_n(C \otimes A) \to \text{ker}(\text{in}_{m-1} \otimes A) \to 0$
Using (5) we see
\[ \ker (i_n \circ \partial) = H_n (C) \cap A \]
\[ \ker (i_{n-1} \circ \partial) = \text{Tor} \left( H_{n-1} (C), A \right) \] QED.
The proof of Thm 3.2 is similar.

**Example**
\[ C : \quad \begin{array}{ccc}
C^0 & \to & C^2 \to C^3 \\
0 & 1 & 2 & 3
\end{array} \]

\( C \) is related to \( \text{SO}(3) = \mathbb{RP}^3 \), i.e.,
\[ H_n (C) = H_n (\mathbb{RP}^3) \]
\[ H_i (\mathbb{RP}^3) = H_i (C) = \begin{cases} 2^{i-1} & i = 0, 1, 2 \\ 0 & i = 3 \end{cases} \]
\( CO \otimes 2/2 : \ \frac{3}{2} \leq \varphi \leq \frac{5}{2} \leq \frac{7}{2} \leq \frac{9}{2} \leq \frac{1}{2} \)

\( H_i(C \otimes 2/2) = \frac{3}{2} \) for \( 0 \leq i \leq 3 \).

\[
\begin{array}{cccccc}
1 & 0 & 1 & 2 & 3 \\
\text{Hom}(C, \frac{3}{2}) & \frac{3}{2} & \frac{3}{2} & 0 & \frac{3}{2} \\
\text{Hom}(C, \frac{3}{2}) & 0 & \frac{3}{2} & 0 & 0 \\
\end{array}
\]

This gives us the same value as above:

\( \text{Hom}(C, \frac{3}{2}) \):
\[
\begin{array}{cccccc}
\frac{3}{2} & 0 & \frac{3}{2} & 0 & \frac{3}{2} \\
0 & 1 & 2 & 3 \\
\end{array}
\]

\( H^i(\text{Hom}(C, \frac{3}{2})) = \frac{3}{2} \) for \( 0 \leq i \leq 3 \).
This agrees with direct calculation.