Thm 3.3.3 \hspace{1em} (Künneth Thm) Let $C$ and $C'$ be chain complexes of free abelian groups. Then there is a SES

$$0 \to \bigoplus_{i=0}^{n} H_i(C) \otimes H_{n-i}(C') \to H_n(C \times C') \to \bigoplus_{i=0}^{n-1} Tor(H_i(C), H_{n-1-i}(C')) \to 0$$

Proof: More of the same.

Example C = C':

\begin{align*}
H_*(C) &= H_*(\mathbb{R}P^2) \\
H_i(C) &= \begin{cases} 0 & i \neq 0, 2, 3 \\
\mathbb{Z} & i = 0, 2, 3 \\
\mathbb{Z}/2 & i = 1 \\
\mathbb{Z}/2 & i = 2 \\
\end{cases}
\end{align*}
**Table of $H_i(C \otimes H_j C)$**

<table>
<thead>
<tr>
<th>$i$</th>
<th>0</th>
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**Table of $T_{ij}(H_i C, H_j C)$**

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Part of $H_2(C \otimes C)$

Part of $H_3(C \otimes C)$

Part of $H_4(C \otimes C)$

What about the homology of a Cartesian product of spaces? Given spaces $X$ and $Y$, we have...
singular chain on $C(x)$ and $C(y)$.

We know how to calculate $\tilde{H}_n(C(x) \otimes C(y))$

in terms of $\tilde{H}_n(x)$ and $\tilde{H}_n(y)$.

But $\tilde{H}_n(x \times y) = \tilde{H}_n C(x \times y)$

$C(x) \otimes C(y) \neq C(x \times y)$

One can construct a chain map

$C(x) \otimes C(y) \to C(x \times y)$

and show it induces an iso in $\tilde{H}_n$

but the proof is very boring.

We will replace the singular chain
complex $S^r$ by a much smaller complex $C[X]$ when $X$ is a CW-complex.

Most geometric objects (manifolds, algebraic varieties, $\mathbb{R}$ on $\mathbb{C}$, furniture, etc.) are CW-complexes. Mapping spaces between such objects are usually not CW-complexes.

Thus (Milnor 1960) let $X$ and $Y$ be CW-complexes. Then the Map $(X,Y)$ (the space of cont. maps $X \to Y$ in
the compact-open topology is always homotopy equivalent to a CW-complex. Let $A_1, A_2, \ldots, A_m \subseteq X \times \mathbb{R}$ and $B_1, \ldots, B_m \subseteq Y$.

Map \((X, A_i), (Y, B_i)) \subseteq \text{Map} (X, Y)

\[ \mathcal{F} \subseteq X \times Y : f(A_i) \subseteq B_i \text{ for } 1 \leq i \leq m. \]

This is also info equivire to a CW-complex. CW-complex is all you need.