Let $X$ be a CW-complex. Its cellular chain complex $C(X)$ is defined by

$$C_n(X) = \text{free abelian group on the set of } n\text{-cells of } X$$

Let $X^n = n\text{-skeleton of } X$

$$X^n/X^{n-1} = S^n \vee S^n \cdots \vee S^n$$

with one wedge summand for each $n\text{-cell}$

The $(n+1)$-cells are attached via some
maps \( S^n \to X^n \to X^n / X^{n-1} \)

For each \( a \) we get a map \( H_n(S^n) \to H_n(VS^n) \)

Taking the direct sum over all \((n+1)-cells\),

we get a map \( C_{n+1}(X) \to \Sigma C_n(X) \)

It is the boundary operator \( \partial_{n+1} \).

Example 1 \( X = CP^m \) - space of complex lines

A point in \( CP^m \) is given by \([z_0, \ldots, z_m]\)
where \((z_0, \ldots, z_m) \neq 0 \in \mathbb{C}^{m+1}\)

For \(a \in \mathbb{C} - \{0\}\) then

\[
[\begin{bmatrix} z_{20} \\ \vdots \\ z_{2m} \end{bmatrix} = a \begin{bmatrix} z_{20} \\ \vdots \\ z_m \end{bmatrix}]
\]

For \(i \leq m\), \(\mathbb{C}P_i^i\) is the subspace defined by \(\mathbb{R}_i = 0\) for \(j > i\).

One gets from \(\mathbb{C}P_i^i\) to \(\mathbb{C}P_{i+1}^{i+1}\) by attaching a \((2i)\)-cell.

CLAIM: \(\mathbb{C}P_i^i - \mathbb{C}^{i+1} = \left\{ \begin{bmatrix} z_0, z_1, \ldots, z_i, 0, \ldots, 0 \end{bmatrix} : z_i \neq 0 \right\} \)

\[
= \left\{ \begin{bmatrix} z_0, z_1, \ldots, z_{i-1}, 1, 0, \ldots, 0 \end{bmatrix} \right\}
\]

\(\times \mathbb{C}^i\).
$CP^i$ is obtained from $CP^{i-1}$ by attaching an $(2i)$-cell. The attaching map $\Sigma^{2i-1} \xrightarrow{b^i} CP^{i-1}$ = set of lines through \(0 \in C^i\) in $C^i$ set of unit vectors in $C^i$

\[f_i(z_0, \ldots, z_{i-1}) = [z_0, \ldots, z_{i-1}]\]

For $X = CP^m$, $C_i = SO$ for $i$ odd

\[C_i = \begin{cases} 2 & \text{for } i \text{ odd} \\ \mathbb{Z} & \text{for } i \text{ even} \end{cases}\quad \text{ and } 0 \leq i \leq 2m\]

and boundary operator is trivial. This means $H^*_i(X) = C_i \cdot (X)$ = as above.
(2) $X = \mathbb{R}P^m$, real projective space.

Similar discussion:

$C_i(x) = \begin{cases} 2^i & \text{for } 0 \leq i \leq m \\ 0 & \text{if } i > m. \end{cases}$

CLAIM: $\Theta_i = \text{multiplication by } 1 + (\mathbf{d})^i$

$= \begin{cases} 0 & \text{for } i \text{ odd} \\ \mathbf{2} & \text{for } i \text{ even} \end{cases}$

Assuming this is true,

$H_i(\mathbb{R}P^m) = \begin{cases} 2^i & \text{dim } \mathbb{O} \\ \mathbf{2} & \text{for } i \text{ odd and } i \leq m \\ 0 & \text{if } i = m \text{ if } m \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$
\[ \mathbb{RP}^2 \quad \pi_0 \mathbb{RP}^2 \cong \mathbb{Z}^2 \]

\[ \mathbb{RP}^3 \quad \pi_0 \mathbb{RP}^3 \cong \mathbb{Z}^2 \cong \mathbb{Z}^2 \]

A proof of the claim not found in the book.

We have a covering \( S^n \to \mathbb{RP}^n \)

which is 2 to 1.

Think of the gp \( C_2 \) acting on \( S^n \)

with \( f(x) = -x \) for a generator \( y \in C_2 \).

The orbit space is \( \mathbb{RP}^n \).

I will define a CW-structure

on \( S^n \) with 2 cells in each.
dimension of to m.

0-skeleton = \mathcal{K}^0 \cong S^0

i-skeleton = S^i \hookrightarrow S^m

(x_0, \ldots, x_i) \mapsto (x_0, \ldots, x_i, 0, \ldots, 0)

The two (i+1)-cells are centered at the points (0, \ldots, 0, \pm 1, 0, \ldots, 0)

We get a cellular chain \varepsilon x with
\[ C_i^r(X) = \begin{cases} 2^2 & \text{for } 0 \leq i \leq m \\ 2 \mathbb{Z} & \text{for } i > m \end{cases} \]

\[ X = \mathbb{Z}^m \]

\[ \forall 0 \leq i \leq m, \quad i > 0 \]

\[ r = 2 \mathbb{Z} \mathbb{Z}_2 = \text{group ring of } \mathbb{Z}/(2) \]

\[ = 2 \mathbb{Z}[x]/(x^2 - 1) \]

\[ = 2 \mathbb{Z}[1, x^2] \]

\[ C^r_\infty(X) \text{ is a chain } C^r \text{ of } R\text{-modules, i.e. } \partial_i \text{ is } R\text{-linear.} \]
Since $C(S^m)$ is a cellular chain, for $S^m$ we must have
\[ H_i(C(S^m)) = H_i(S^m) = \{ \mathbb{Z} \text{ for } i = 0, m \} \{ 0 \text{ otherwise} \} \]

There is a $C_2$-equivariant map
\[ S^m \rightarrow \text{pt.} \ (\text{with trivial } C_2\text{-action}) \]
\[ C_i(S^m) \rightarrow C_i(\text{pt}) = \{ \mathbb{Z} \text{ for } i = 0 \} \{ 0 \text{ for } i > 0 \} \]

There are 2 different $R$-module structures on $\mathbb{Z}$ corresponding to the 2 actions of $C_2$ on $\mathbb{Z}$.
with \( \chi(1) = \pm 1 \)

Trivial action \( \mathbb{R} / (\chi-1) = \mathbb{Z}_+ \)

Nontrivial action \( \mathbb{R} / (\chi+1) = \mathbb{Z}_- \)

Recall \( \mathbb{R} = \mathbb{Z}[\chi] / (\chi-1)(\chi+1) \)

(not an integral domain)

Claim there is a unique boundary linear operator on \( C(S^m) \) that gives
\( \mathbb{H} \circ C(S^m) = \mathbb{H} \circ S^m \)

with \( \mathbb{H}_0 = \mathbb{Z}_+ \)
$H_6 = \text{ker} (1+x) \subseteq R$

$= (1-x)$  

Note $\varphi (1-x) = x - x^2 = x(1-x) = -(1-x)$

$= \mathbb{Z}_+$

$H_6 = \mathbb{Z}_+$

How to pass from $C(S^m)$ to $C(\mathbb{R}P^m)$? ?