$S^3 \to S^2 \to S^2 \vee \mathbb{C}P^2 = \mathbb{C}P^2$ HOPF map

$S^3 \to S^2 \to S^2 \vee S^4 = W$ trivial map

$H^* W$ and $H^* \mathbb{C}P^2$ are the same as graded abelian groups but not as graded rings

Given an arbitrary map $S^3 \to S^2$, we can form its mapping cone $C_0 > S^2 \vee C S^3 = S^2 \vee \mathbb{C}P^2$.
\[ H^i(G, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2, 3, 4 \\ 0 & \text{else} \end{cases} \]

Let \( \chi \in H^2 \) and \( y \in H^4 \) be generators. Then \( \chi^2 = h(g) y \in H^4 \) for some \( h(g) \in \mathbb{Z} \).

\( h(g) \) is defined up to sign.

\( h(g) \) is called the Hopf invariant of \( g \). It defines a map

\[ \pi_2 \colon S^2 \to \mathbb{Z}, \quad \text{It is in fact an isomorphism.} \]
A generalization:

\[ s^{4n-1} \rightarrow s^{2n} \quad C_g = s^{2n} \cup e^{4n} \]

\[ H^i C_g = \begin{cases} 2 & \text{for } i = 0, 2n, 4n \\ 0 & \text{else} \end{cases} \]

We can define the Hopf invariant \( h(g) \) as above.

Remark: Consider

\[ s^{4n-3} \cup s^{2n-1} \]

for some \( n \geq 1 \)

\[ C_g = s^{2n-1} \cup e^{4n-2} \]

\[ H^i C_g = \begin{cases} 2 & \text{for } i = 0, 2n-1, 4n-2 \\ 0 & \text{else} \end{cases} \]
Let \( x \in \mathbb{Z}^{n+1} \) and \( y \in \mathbb{Z}^{4n-2} \) be generators. Because of the sign issue, \( x^2 = -x^2 \in \mathbb{Z}^{4n-2} \), hence \( x^2 = 0 \) and we get no information about \( g \).

Back to \( S^{4n-1} \rightarrow S^{2n} \). We get a hom \( \pi_{4n-1} : S^{2n} \rightarrow \mathbb{Z} \). It is not an isomorphism for \( n \geq 1 \).

Question: What is its image? Two known facts: 
(1) For $n = 1, 2, 4$, it is onto, i.e., a map with Hopf invariant 1. They were constructed by Hopf in 1930.

For $n = 2$, replace $\mathbb{C}$ by the quaternion $\mathbb{H}$, a noncommutative division algebra additively isomorphic to $\mathbb{R}^4$.

For $n = 4$, replace $\mathbb{C}$ by the octonions $\mathbb{O}$, a nonassociative division algebra isomorphic to $\mathbb{R}^8$. 

CAYLEY NUMBERS
(2) For all $n > 0$, there is a map $S^{4n-1} \lor S^{2n} \to S^{2n}$ with Hopf invariant 2.

Consider $S^{2n} \times S^{2n}$. It has a CW structure with two $2n$-cells and one $4n$-cell. The attaching map for the latter is $S^{4n-1} \lor S^{2n} \to S^{2n} \lor S^{2n}$.

\[ g \quad \text{fold} \quad S^{2n} \]

Claim: $H(g) = 2$. 
let $x$ and $y$ be the gens of $H^{2n}(S^{2n} \times S^{2n})$. Then $H^{4n}(S^{2n} \times S^{2n}) = \mathbb{Z}$ is generated by $xy$. Can show $x^2 = y^2 = 0$ using the maps $S^{2n} \times S^{2n} \xrightarrow{p_1 \times p_2} S^{2n}$ where $x \in \text{Im } p_1^*$ and $y \in \text{Im } p_2^*$.
Hence the same is true of $x^2$ and $y^2$ and $H^{4+n} S^{2n} = 0$. This determines $H^* (S^{2n} \times S^{2n})$ as a graded ring.

\[ x + y \]

\[ S^{2n} \vee S^{2n} \xrightarrow{i} C_b = S^{2n} \times S^{2n} \]

\[ \text{fold} \]

\[ S^{2n} \xrightarrow{j} C_g = S^{2n} \cup g \cdot e^{4-n} \]

\[ \text{fold} \]

\[ (x+y)^{2n} \]

\[ w^{2n} \cup w^{2n} \wedge w^{2} \cup w^{2} \cdot e^{4-n} \]

The fold map induces an iso in $H^{4+n}$.
In \( H^4(S^{2n} \times S^{2n}) = \mathbb{Z} \) generated by \( xy \)

\[(x+y)^2 = x^2 + 2xy + y^2 = 2xy\]

This implies \( H_4(g) = 0 \).

Hence we have a lower bound on the image of \( \tilde{H}_{4n-1}(S^{2n}) \rightarrow \mathbb{Z} \).

It is onto for \( n = 1, 2, 3, 4 \).

It has index \( \leq 2 \) for other \( n \).

Thm (J. F. Adams 1961) There is no map of Hopf invariant 1.
for $n \neq 1, 2, 4, 8$. DEEPCON: There is no division algebra structure on $\mathbb{R}^m$ for $m \neq 1, 2, 4, 8$.

New topic: Poincaré Duality

Recall a $n$-manifold $M^n$ is a topological space in which each point has a nbd $\cong \mathbb{R}^n$. A manifold with boundary is $\mathbb{R}^n$.
a smooth \( M \) in which each \( x \in M \) is as above \( \mathbb{R} \) has a nbhd homeo to "1/2 of \( \mathbb{R}^n \)"

\[
\mathbb{Q} = \{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n = 0 \}\]

A manifold is closed if it is compact and has no boundary.

Let \( M \) be a closed path connected \( n \)-mfd. It is a CW-ax with cells in dim \( \leq n \).

Known examples:
\( \pi_i, i \geq 0 : S^n \times S^n \) surface of genus \( g \),

\( \mathbb{C}P^{n/2}, \mathbb{R}P^n \), Klein bottle, \( n \) even.

In each case \( H_n(M) = \mathbb{Z} \) or \( 0 \),

e.g., \( H_2 \mathbb{R}P^2 = 0 \) and \( H_2 KB = 0 \).

This is related to orientability. \( \mathbb{R}P^2 \) and \( KB \) are not orientable.

Let \( M^n \) be a closed orientable path, \( m \).
Thm (Poincare duality)

1) \( H^i(M,\mathbb{Z}) \cong H_{n-i}(M,\mathbb{Z}) \)
e.g. \( H^0(M,\mathbb{Z}) = \mathbb{Z} \). This implies \( H_m(M,\mathbb{Z}) = \mathbb{Z} \) via UCT.

2) Let \( x \in H^i(M) \) \( B \in H^j(M) \) with \( i + j \leq n \). \( \alpha \beta \in H^{i+j}(M) \cong H^{n-i-j}(M) \).

We have elements \( \alpha \in H_{n-i}(M) \) and \( \beta \in H_{n-j}(M) \).

Suppose there are represented by submanifolds of dimo \( n-i \) and \( n-j \) (codimensions \( i \) and \( j \)?)
Then their intersection refe
an element, \( c \in H_{n-1-y}(M) \)
\( c \) is Poincaré dual to \( \alpha \beta \)