Recall a $\Delta$-complex $X$ is a space obtained by gluing simplices together. To such a space we associated a chain complex $C(X)$ where $C_n(X)$ = free ab gp on the n-simplices.

We can calculate its homology.

This approach has its limitations.

Singular homology, the real stuff.

$X$ = topological space
Def. A singular n-complex in X is a contr map \( \Delta^n \rightarrow X \).

The singular chain complex \( S(X) \) is defined by

\[
S_\cdot(X) = \text{free abelian gp on all singular n-simplices}.
\]

To define its boundary operation

\[
S_\cdot(X) \xrightarrow{\partial} S_{\cdot - 1}(X)
\]

\[
\partial^\Delta_n : \bigoplus \Delta_{n,i} \rightarrow \Delta^n \xrightarrow{\partial} X.
\]

\[
Q_n(X) = \bigoplus_{0 \leq i \leq n} (-1)^i \sigma_i \Delta_{n,i}.
\]
The same calculation as before shows $Q_{n-1}Q_n = 0$, so $S(x)$ is a chain $\mathcal{X}$. $H_*(X) = H_*(S(X))$.

This is the main definition of the course.

Example $X = pt$. Then $S_m(x) = 2$ for all $m \geq 0$.

Since $\exists$ one map $\Delta^n \to pt$.

$S_0(x) \rightarrow S_1(x) \rightarrow S_2(x) \rightarrow S_3(x) \rightarrow S_4(x) \rightarrow \cdots$

We find $S_{n+1}$ is the same for all $i$. 
\[
H_n(X) = \begin{cases} 
2 & \text{for } n = 0 \\
0 & \text{for } n > 0
\end{cases}
\]

This is the Dimension Axiom

Then let \( X = \bigsqcup_{\alpha} X_\alpha \) where each \( X_\alpha \) is path connected and both closed and open in \( X \).

Then \( S(X) = \bigsqcup_{\alpha} S(X_\alpha) \) and

chain complexes, \( \varphi \)

\[
H_\ast(X) = \bigoplus_{\alpha} H_\ast(X_\alpha)
\]
Proof: Any map $\Delta^n \to X$ lands in one $X_\alpha$, so $S_m(x) = \bigoplus \delta \cdot S_m(x_\alpha)$ and $\Theta_m$ respects this splitting. QED

Prop 2.7 Let $X$ be path-connected. Then $H_0(X) = \mathbb{Z}$. 

Proof: Consider $\xi \in \pi_0(X) \cong \mathbb{Z}$. Let $\xi$ be defined later.

$S_0(X) = \text{free ab gp on the set } X$.

For each $x \in X$ we define $\xi(x) = 1$, where $x$ is the generator of $S_0(x)$ corresponding to $x$. 
\[ H_0(X) = \frac{S_0(X)}{\text{im } \Theta_1} \] by definition.

Claim: \( \text{im } \Theta_1 = \ker \varepsilon. \)

\[ S_1(X) = \text{free abgp on } \Delta^1 \to X \]

\[ = \text{free abgp on paths } I \to X. \]

\[ \Theta_1(p) = \beta(0) - \beta(1) \in S_0(X). \]

\[ \varepsilon \Theta_1(p) = 1 - 1 = 0 \]

So \( \text{im } \Theta_1 \subset \ker \varepsilon. \)

Exercise: \( \ker \varepsilon \subset \text{im } \Theta_1 \)

It follows that

\[ S_1(X) \xrightarrow{\varepsilon} S_0(X) \xrightarrow{\Theta_1} I \to 0 \]

is exact, so \( S_0(X)/\text{im } \Theta_1 = \mathbb{Z}. \)
Con: For \( X = \coprod \alpha X_\alpha \) as above then
\[
H_0(X) = \bigoplus \alpha \geq \text{free ab gp on path components.}
\]

Homotopy Axiom:

If \( f, g : X \to Y \) are homotopic then \( H_*(f) = H_*(g) \). We need to construct a chain homotopy between \( S(f) \) and \( S(g) \). This is a collection...
of home $\Sigma_n(X) \xrightarrow{\delta} \Sigma_{n+1}(Y)$ with certain properties. Let $f : X \times I \to Y$ be a homotopy $f$ and $g$, and let $\Delta^n \to X$ be a singular $n$-simplex. Given $\Delta^n \times I \xrightarrow{\delta} X \times I \xrightarrow{f} Y$, $\Delta^n \times I \xrightarrow{\delta} X \times I \xrightarrow{g} Y.$