Prop 2.21  For an open covering $U$ there are \( C^U(X) \xrightarrow{\subset} C(X) \).

\( L \) is the obvious inclusion and \( \varphi \) will be defined later.

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small simplices suffice
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Proof. Define the subdivision operator

\[
C^n(X) \xrightarrow{S} C^{n+1}(X)
\]

\( 6 \rightarrow \) sum of \((n+1)\)! smaller simplices from nonconstant subdivision
This can be shown to be a chain homotopy equivalence (see pages 121–123).

Let $U$ be an open cover of $X$.

Define $\Delta^n \hookrightarrow X$, we get an open covering of $\Delta^n$ by the $U_i$. Since $\Delta^n$ is compact, it is covered by finitely many such open sets. If we subdivide $\Delta^n$ $m(\delta)$ times (for $m(\delta)$ large enough) and each little simplex is contained in some $U_i$. 
Let \( C_n(x) \xrightarrow{D_m} C_{n+1}(x) \) be a chain homotopy between \( S_m \) and \( 1_c(x) \).

Want to define \( C(x) \xrightarrow{D} C(U(x)) \). We will do so by defining a chain homotopy \( S \) between \( p \) and \( 1 \).

\[
\begin{align*}
C_n(x) & \xrightarrow{D_m} C_{n+1}(x) \\
(\Delta \circ x) & \xrightarrow{D_m(x)} D_m(x)
\end{align*}
\]

\( D \) determines the chain map \( p \).
To prove 2.11 we need to show both
1. \( C^U(X) \xrightarrow{\varepsilon} C(X) \xrightarrow{\varphi} C^U(X) \)
and 2. \( C(X) \xrightarrow{\varphi} C^U(X) \xrightarrow{\eta} C(X) \)
are chain homotopic to the identity,
1. \( \varphi \) is the identity on \( C^U(X) \).
2. \( \varepsilon \) is a chain homotopy between \( L \varphi \) and \( 1_{C(X)} \).

QED.

**Historical note**

By 1910 spaces had “Betti numbers.”
For a space \( X \), \( B_n(X) \) is the # of free abelian summands in \( H_n(X) \).
e.g. for a surface $X$ of genus $g$, $\beta_1(X) = 2g$.

**Question:** How to describe $H_*(X \times Y)$ in terms of $H_*(X)$ and $H_*(Y)$?

**Theorem:** $\pi_n(X \times Y) = \pi_n(X) \oplus \pi_n(Y)$ for path-connected $X$ and $Y$.

**Sketch of proof:**

This is a pullback diagram:

$$
\begin{array}{ccc}
X \times Y & \to & Y \\
\downarrow & & \downarrow \\
X & \to & X
\end{array}
$$
i.e. given maps $f, g \in \mathcal{F} \setminus \mathcal{H}$ ,

$\exists h: W \to X \times Y$

making the diagram commute

$h(w) = (f(w), g(w))$

For $W = S^n$, $f, g$ and $h$ represent elements

in $\pi_n(X)$, $\pi_n(Y)$ and $\pi_n(X \times Y)$.

The third is uniquely determined by the first + second. QED

However $\pi_n(X \times Y) \neq \pi_n(X) \otimes \pi_n(Y)$

in general, e.g. $X = Y = S^3$, $n = 2$. 
$$H_2(S^1) = H_2(S^4) = 0$$

but $$H_2(S^1 \times S^4) = H_2(\text{torus}) \neq 2$$

Related question:

Given two chain complexes $$C'$$ and $$C''$$, describe $$H_\bullet(C' \otimes C'')$$ in terms of $$H_\bullet C'$$ and $$H_\bullet C''$$.

To define $$C = C' \otimes C''$$

$$(\sum a_n x_i^i)(\sum b_n x_j^j) = \sum (\sum a_i b_n) x_{i+j}$$

$$C_n = \bigoplus_{i+j=n} C_i \otimes C_j$$
When $x \in C_i^j$ and $y \in C_i^{j'}$ what is $\vartheta(x \otimes y) = \vartheta'(x) \otimes y + x \otimes \vartheta''(y) \otimes \\
\in C_i^{j'} \otimes C_i^{j''} \otimes C_i^{j'''}$ \\
\subset C_i^{j''''}$

Try this: $\vartheta_2(x \otimes y) = \vartheta_2'(x) \otimes y + x \otimes \vartheta_2''(y) \otimes \\
\in C_i^{j-1} \otimes C_i^{j''} \otimes C_i^{j'''} \otimes C_i^{j''''}$

Then $\vartheta_2(x \otimes y) = \vartheta_2'(x) \otimes y + x \otimes \vartheta_2''(y) \otimes \\
= \vartheta_2'(x) \otimes y + \vartheta_2(x) \otimes \vartheta_2''(y) \otimes \\
= 2 \vartheta_2'(x) \otimes y + \vartheta_2(x) \otimes \vartheta_2''(y) \\
+ \vartheta_2'(x) \otimes \vartheta_2''(y) + x \otimes \vartheta_2''(y) \otimes \\
= 2 \vartheta_2'(x) \otimes \vartheta_2''(y) \neq 0 \text{ in general.}$
TAKE 2: \( \mathcal{E}(x \otimes y) = \mathcal{E}(x \otimes y + (i)^2 x \otimes \mathcal{E}''(y)) \)

Then \( \partial \mathcal{E} = 0 \).