Question: What is $H_*(X \times Y)$ in terms of $H_*(X)$ and $H_*(Y)$?

Related algebraic question: Given 2 chain complexes $C'$ and $C''$ with $C = C' \oplus C''$, what is $H_*(C)$ in terms of $H_*(C')$ and $H_*(C'')$?

Naive guess: $H_n(C) = \bigoplus_{0 \leq i \leq n} H_i(C') \oplus H_{n-i}(C'')$

NOT TRUE IN GENERAL

Easier question: For a chain $C$ and abelian group $A$, what is $H_*(C \otimes A)$?
In general it is not $H_*(C) \otimes A$.

Another similar question: What about $\text{Hom}(C, A)$?

\[
\begin{align*}
C_0 & \xrightarrow{\partial_1} C_1 \xrightarrow{\partial_2} C_2 \xrightarrow{\partial_3} C_3 \\
\end{align*}
\]

$\text{Hom}(C, A) \xrightarrow{\delta^0} \text{Hom}(C_1, A) \xrightarrow{\delta^1} \text{Hom}(C_2, A) \xrightarrow{\delta^2} \ldots$

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Diagram:
\[
\begin{array}{c}
\text{Hom}(C, A) \\
\text{Cochain Complex}
\end{array}
\]
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$\text{Hom}(C_0; A) \xrightarrow{\delta^0} \text{Hom}(C_1, A) \xrightarrow{\delta^1} \text{Hom}(C_2, A) \xrightarrow{\delta^2} \ldots$

Define $C_m \xrightarrow{d^m} A$,

\[
\begin{align*}
2m+1 & \quad C_m \\
2m+2 & \quad C_{m+1}
\end{align*}
\]

The "homology" of the above is denoted by $H^\ast \left( \text{Hom}(C, A) \right)$. 

It is not $\text{Hom}(H_x(C), A)$ in general.

Trying to prove $H_x(C \oplus A) = H_x(C) \oplus A$
on $H^x(\text{Hom}(C, A)) = \text{Hom}(H^x(C), A)$ fails because

The functors $(\cdot \oplus A)$ and $\text{Hom}(\cdot, A)$ do not preserve exactness, i.e., given a short exact sequence

\[ 0 \to B' \to B \to B'' \to 0 \]

then $0 \to A \oplus B' \to A \oplus B \to A \oplus B'' \to 0$.
\[ 0 \to \text{Hom}(B', A) \to \text{Hom}(B, A) \to \text{Hom}(B'', A) \to 0 \]

and

\[ 0 \to \text{Hom}(A, B') \to \text{Hom}(A, B) \to \text{Hom}(A, B'') \to 0 \]

are NOT exact in general.

Example

\[ 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0 \]

and

\[ A = \mathbb{Z}/2, \quad B = \mathbb{Z}, \quad B'' = \mathbb{Z}/2 \]

Tensoring \((\star)\) with \(A\) gives

\[ 0 \to \mathbb{Z}/2 \to \mathbb{Z} \to \mathbb{Z}/2 \to 0 \]

NOT \(-1\)

Apply \(\text{Hom}(\star, A)\) to \(\star\) gives

\[ 0 \to \mathbb{Z}/2 \to \mathbb{Z} \to \mathbb{Z}/2 \to 0 \]
Apply \( \text{Hom}(A, \cdot) \) to \( x \) gives:

\[
0 \to 0 \to 0 \to \mathbb{Z}/2 \to 0
\]

What to do about this? ??

**Answer:** Homological algebra.

Define an \( R \)-module \( P \) is projective if whenever \( P \), \( \varphi \in \text{Hom}(M, N) \), \( N \), \( M \) are \( R \)-modules and \( \varphi \) is onto,

\[
M \xrightarrow{\alpha} N \xrightarrow{\varphi} 0
\]

fix with \( \beta = \alpha \varphi \).

Prop: A free \( R \)-module is projective.
e.g. suppose \( P = R \), \( B \) is determined by \( B(1) \), which could be any element in \( N \). Define \( X(1) \) to be some elt in \( M \) mapping to \( B(1) \).

Def. A projective resolution of an \( \mathbb{R} \)-module \( M \) is a long exact sequence of the form

\[
0 \rightarrow M \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow \cdots
\]

where each \( P_i \) is projective.

Examples: \( \mathbb{R} \)-field. Every module \( M \) is free, so we can define
\[ P_i = \begin{cases} S & \text{for } i = 0 \\ 0 & \text{for } i > 0 \end{cases} \]

**Definition:** A ring \( R \) is a **principal ideal domain (PID)** if it is an integral domain (no zero divisors) and each ideal \( I \subseteq R \) is generated by a single element.

**Example:** \( \mathbb{Z} \) is a PID.

Any **field** is a PID.

\( \mathbb{Z}[x] \) is not a PID e.g. \((2, x)\).
Thm. If $M$ is a module over a PID $R$, it has a form of the form $0 \subseteq M \subseteq P_1 \subseteq \cdots$.

Thm. If $M$ is a finitely generated module over a PID $R$, then $M$ is a direct sum of cyclic $R$-modules.

e.g. $\mathbb{Q}$ is not fin. gen. over $\mathbb{Z}$ and is not a sum of cyclic $\mathbb{Z}$-modules.
Then every $R$-module $M$ has a free resolution.

If we can find a projective module $P_0 \to M \to 0$

e.g. $P_0$ could be the free $R$-module
generated by the elements of $M$.\[0 \to M \xrightarrow{i_0} P_0 \xrightarrow{i_1} M_1 \to 0\]

Let $P_2$ be a free $R$-module mapping onto $M_1$. Then we have
\[0 \to M \xrightarrow{i_0} P_0 \xrightarrow{i_1} P_1 \xrightarrow{i_2} P_2 \to 0\]

$M_2 = \ker i_1$, $k_2$
Let $P_2 \mapsto M_2$ be onto with $P_2$ proj. etc. QED.