Vector field problem: How many linearly independent tangent vector fields does $S^n$ have?

For $n+1 = x \cdot 2^k$ for $x$ odd, then $S^n$ has $\phi(x)$ where

$$\phi(x) = \begin{cases} 0 & \text{for } x = 1, \\ 2 & x = 2, \\ 3 & x = 3, \\ 8 + \phi(x-4) & x \geq 4 \end{cases}$$

There were constructed ~1930.

Are there any more? ???
$S^n$ has $j$ vector fields iff $TS$ with $p_0=1$.

iff $TS'$ with $p's'=1$, $S^n$. We know $TS'$ for $i > \phi(k)$, Does it exist for $i = 1 + \phi(k)$?
Can show 

\[ i ; w . s \] is 
null.

\[ 0 \Rightarrow \]

\[ i \Rightarrow \]

\[ \phi (k) \]

\[ C_i \Rightarrow P_{n-1} \phi (k) \]

\[ P_n \Rightarrow C_i \]

\[ P_{n-1} \phi (k) \]

\[ S_{n-1} \phi (k) \]

\[ t \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \Rightarrow \]

Adams showed 

\[ t = J (x_k) \neq 0 \]

So no 
extra vector 
field exists.

The vertical sequence above is 
a [cofiber sequence], which is 
as follows.

Given \( A \rightarrow B \),
define its mapping cylinder $M_f$

$$M_0 = A \times I \cup B / (a, 1) \sim f(a) \quad M_0 \sim B,$$

$$C_f = M_0 / A \times \{0\} = cofiber \ of \ f.$$
\[ \sum A = A \times \mathbb{I} / (A \times \mathbb{S} \sim \mathbb{A}, A \times \mathbb{B} \sim \mathbb{A}) \]

Def. The \( J \)-homomorphism

\[ \pi_k(O(n)) \to \pi_{n+k} S^n \quad \text{for } k, n > 0. \]

is defined as follows. Given \( S^k \to O(n) \),

we get a map \( S^k \times D^n \to D^n \)

Extend \( g \) to all of \( S^{n+k} \) by sending the complement of \( S^k \times D^n \) to the basept.
Facts about $\prod_{k=0}^{n-1} SO(2k+1)$

1) both groups are independent of $n$ if $n > k+1$.

2) Batts then describes the source of

$$
\prod_{k=0}^{n-1} SO = \begin{cases} 
2 & \text{if } j \equiv 3 \text{ or } 7 \mod 8 \\
2/2 & j \equiv 0 \text{ or } 1 \\
0 & \text{other cases}
\end{cases}
$$

Let $x_k$ be a generator of $\prod_{k=0}^{n-1} SO$

3) For each $k \geq 0$, $J(x_k)$ then generates a direct summand of $X$
\[ \prod_{n+p(k)} \leq \sqrt{n} \text{ for } n \geq 2. \]

4) We get a summand \( 2/2 \) for \( k = 0, 1 \mod 4 \).

For \( k = 2, 3 \), the order of \( J(xk) \) has an arithmetic description:

<table>
<thead>
<tr>
<th>( k )</th>
<th>2</th>
<th>3</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi(k) )</td>
<td>3</td>
<td>2</td>
<td>11</td>
<td>15</td>
</tr>
</tbody>
</table>

order of \( J(xk) \): 24 | 240 | 504 | forget

The numbers in the bottom row arise in the study of the Riemann
zeta function:

\[ \phi \left( \phi(x) \right) \leq \prod_{n=0}^{\infty} \phi(x^n) \]