Applications of \( \Pi_1(S^1) \rightarrow \mathbb{Z} \)

**Fundamental Theorem of Algebra**
(Dumas 1801). Let \( p(x) \in \mathbb{C}[x] \)
nonconstant. Then \( \exists z \in \mathbb{C} \) with \( p(z) = 0 \).

This means \( p(x) = (x-z) \cdot p_1(x) \)
if degree \( p(x) \) is \( n \), then deg \( p_1(x) = n-1 \)
\( \Rightarrow p(x) = (x-z_1)(x-z_2)\ldots(x-z_m)C_n \)

Topological proof
Let \( \psi(x) = C_n x^n + C_{n-1} x^{n-1} + \cdots + C_0 \)
with \( C_n \neq 0 \). Assume \( C_0 \neq 0 \) and \( C_n = 1 \), \( \psi \) defines a continuous map
\[ C \to S^1. \]
Assume \( 0 \notin \text{Im} \psi \)
\[ C \to \{0\} \]
Let \( f(x) = \frac{\psi(x)}{||\psi(x)||} \in S^1. \]
Will restrict \( f \) to a circle about 0 of radius \( m \) for various \( m \geq 0 \).
\[ f_n(x) = \frac{\psi(ne^{2\pi i x})}{||\psi(ne^{2\pi i x})||} \in S^1. \]
for $0 \leq s \leq 1$. This defines a map $S^1 \to S^1$ representing an element in $\pi_1 S^1 = \mathbb{Z}$. We will show that for small $n$ we get $0$ and for larger $n$ we get $\pi$.

We have $f(z) = z^n + \sum_{i=0}^{n-1} c_i z^i$.

For small $n$ this is close to $c_0 \neq 0$ so we get $f_n$ is null homotopic.

Choose $M > \sum |c_i|$ then $|z^n| > \sum |c_i| z^{|i|}$.
Let $p_t(z) = z^t + x \sum c_i z^i$ for $0 \leq t < 1$.

Note $p_t(z) \neq 0$ for all $t$ and $x$.

Let $p_0(z) = z^n$ and $p_1(z) = p(z)$.

Hence $p(z)$ and $z^n$ define homotopic maps $S^t \to S^n$.

**CONTRADICTION.**  QED.

Brouwer Fixed Point Theorem 1910

Let $D^2 \to D^2$. Then $\exists x \in D^2$
with \( f(x) = x \).

Proof. Assume there is no such \( x \).

Will define a map \( \partial D^2 \to S^1 \).

s.t. \( g \partial D^2 \) is the identity.

We can use \( g \) to construct a null homology of \( S^1 \).

\[ S^1 \times I \to S^1 \]
\[ h(x, t) = g(tx), \quad 0 \leq x \leq 1 \]
\[ m(x, 1) = g(x) = x \]
\[ m(x, 0) = g(0) \in S^n \]

This contradicts \( \pi_1(S^n) = \mathbb{Z} \)

QED.

Remark: If we prove that the identity map on \( S^{n-1} \) is not homotopic to a constant map, it will follow that any map
$D^n \rightarrow D^n$ has a fixed point.

Theorem 1.10 (Borsuk-Ulam). For any $f: S^2 \rightarrow \mathbb{R}$, there exists $x \in S^2$ with $f(x) = f(-x)$.

Proof. Assume $f(x) \neq f(-x)$ for all $x$.

Let $g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|} \leq S^2 \rightarrow S^1$. 

$g(-x) = -g(x) \leq S^2 \rightarrow S^1$. 
Will show there is no map \( g \) as above with \( g(-x) = -g(x) \).

Consider \( \tilde{h}(s, \omega) := \varphi(s) + \omega \) \( \forall 0 \leq s \leq 1 \) \( \tilde{h}(s, \omega) = \tilde{g}(\omega) \) \( \forall \omega \in \varnothing \). Let \( \varnothing \) be the empty set in \( \mathbb{R} \).

\[ h(s) = -h(0) \quad \forall s \in (0, 1) \]

\[ h(s + 1/2) = h(s) + \omega/2 \quad \text{for some} \]
\( h(15) = h(\frac{1}{2}) + \frac{9}{2} = h(0) + \frac{9}{2} \)

\[ \Rightarrow h(0) = \frac{9}{2} \]

This means \( h \) defines a \( +0 \)-elt in \( \pi_1(S^4) \). The equation is essential (not just to constant), but it extends to the southern hemisphere. 

**CONTRADICTION.** \( \therefore \)
We have shown \( g : S^2 \to S^1 \)
with \( g(-x) = -g(x) \).

**Ham Sandwich Theorem.**

Given 3 compact regions \( K_1, K_2, \text{and} K_3 \) in \( \mathbb{R}^3 \), there is a plane that
bisects all 3.

**Proof:** Each plane has a unit normal vector. Any plane \( P \)
is parallel to one that bisects \( K_1 \).
Assume $P$ intersects $K_1$. It is a unit normal vector in $\mathbb{R}^2$.

$x \in \mathbb{S}^2 \rightarrow \text{plane } P \text{ intersecting } K_1$.

For $K_2$ and $K_3$, we look at the volumes lying above and below $P$.

Hence we get a map $\mathbb{S}^2 \rightarrow \mathbb{R}^2$.

$g(-x) = -f(x)$. We are looking for $x$ with $f(x) = (0,0)$. Assume $\exists$ no such $x$. Define $g(x) = \frac{b(x)}{|b(x)|} \in \mathbb{S}^1$. 
We now have \( g : S^2 \rightarrow S^1 \) with \( g(-x) = -g(x) \). CONTRADICTION

QED

Next: Van Kampen Theorem

Let \( X = A \cup B \) with \( A \cap B, A, B \) and \( X \) all path connected. Let \( x \in A \cap B \) and assume that \( \pi_1(A \cap B), \pi_1(A) \) and \( \pi_1(B) \) are known as are the homomorphisms.
Van Kampen Thm gives a formula for $\pi_1(X)$ in terms of these data.