Let $X$ be a space which is

i) path connected  

ii) locally path connected LPC

iii) semi-locally simply connected SLC

Will construct a simply connected

a) simply connected covering space $\tilde{X}$

Choose a base point $x_0 \in \tilde{X}$

Two paths $\gamma_1, \gamma_2$ starting at $x_0$

are equivalent if
1) \( \chi(1) = \chi'(1) \) (same end point)

2) they are homotopic relative to their end points, i.e. there is an end point preserving map

\[ h : \mathbb{I}^2 \rightarrow X \]

\[ h(0,0) = \chi(0) \]
\[ h(1,0) = \chi'(0) \]
\[ h(0,1) = \chi(1) \]
\[ h(1,1) = \chi'(1) \]
Let $[x]$ denote the equivalence class of $x$. 

$x$ as a set in $\mathcal{E} = \{ [x] : [x] \text{ is defined by } \varphi([x]) = 1 \}$. 

Need to totalize $\mathcal{E}$, show $P$.
is a covering and show $\pi_1 \tilde{X} = 0$.

We will give a basis for the topology of $\tilde{X}$
\[ U = \bigcup \{ U \subseteq \tilde{X} \mid \pi_1(U) \to \pi_1(\tilde{X}) \text{ is trivial} \} \]

It is a basis for the topology of $\tilde{X}$.

Note: A space is convex if each pair of

each $x \in E$ has a convex hull.

Hence it has a convex basis.

Then such an open set $U \subseteq E$ and
a path $\gamma$ from $x_0$ to a point $x_1 \in U$, \\
\[ \forall \gamma : \gamma' \subseteq \gamma : \eta(t) = \gamma(t) \text{ and } \eta(-1) \in U \] \\
Let $\mathcal{U} = \{ U_{x_0} \}$ as above. \\
Claim: This is the basis of a topology.
on $E$. For details see pages 64-65 of Hatcher.

Remark 1) The map $g : \mathbb{R} \to U$ is onto since $U$ is path-connected.
2) It is $1-1$ because any two paths in $E$ would produce homotopic paths in $\mathbb{R}$.

Hence it is a homeomorphism.
To show \( p \) is a covering, we need to show each \( U \in \mathcal{U} \) is evenly covered:

\[
p^{-1}(U) = \bigcup_{x \in \pi^{-1}(U)} \{x \}
\]

Remark: If \( \tilde{U} \) is \( U \)-invariant, then \( \tilde{U} = \tilde{U}_{x_0} \) and if \( \pi = \tilde{f} \circ \tilde{r} \) then

\[
\tilde{U}_{x_0} = \bigcup_{n \in \mathbb{Z}} \{ x_0 \circ n \} \text{ in } \tilde{U}
\]

\[
\left\{ x \circ n, \circ n' : n, n' \in \mathbb{Z}, \text{ in } \tilde{U} \right\}
\]
Claim: $f^{-1}(U) \times V \subset D$ where

Hence $p$ is a covering

Need to show $\hat{X}$ is simply connected

We know $\hat{X} \sim \mathbb{R}$, so it suffices to show it has trivial image. An element in the image is represented by a closed path $\gamma$ at $x_0$ in $X$ that lifts to a closed path
\( \hat{x} \) in \( \hat{X} \) at \( \hat{x}_0 \) (the homotopy class of the constant path at \( x_0 \)). Let 
\( \gamma_t = \hat{x} | [0, t] \) for \( 0 \leq t \leq 1 \).

The path \( \hat{y} \) in \( \hat{X} \) is defined by 
\( \hat{y}(t) = [\gamma_t] \in \hat{X} \).

If \( \hat{y} \) is closed, then \( \hat{y}(1) = \hat{x}_0 = [x_0] \).

But \( \hat{y}(1) = [x_1] = [x_0] \). This means \( [x_1] = [x_0] \), i.e., \( x_1 \) is null homotopic. This means
the hom \( \tilde{\pi} : \tilde{X} \to \pi_1 X \) is trivial, so \( \tilde{X} \) is simply connected. QED

**Proof 1.35** Let \( H \subset G \) and \( \tilde{X} \) be as above. Then there is a path conn covering \( \tilde{X}_H \to \tilde{X} \) with \( \tilde{\pi} : (X, H) = H \).

Proof. Recall in the previous proof, two paths \( x \) and \( x' \) were equivalent if \( x(1) = x'(1) \) and there is an endpoint
preserving homotopy between them. Equivalently, the closed path \\
\gamma \circ \tilde{F}' is null homotopic to 0.

We can modify this condition to: \gamma \circ \tilde{F}' is a closed path representing an element in H. Then we can argue as before, i.e. \tilde{\Sigma}_H is the set of such equivalence classes, and so on.
There is a bijection between path connected coverings of $X$ and subgroups of $\pi_1(X)$. 

Homology.

Recall a space $X$ with base point $x_0$ has homotopy groups $\pi_n(X, x_0)$ for each $n \geq 1$, which is
abelian for $n \geq 2$.

$\pi_0(X)$ = set of path connected components of $X$

It has no natural group structure.

In each case a map $(X,x_0) \to (Y,y_0)$ leads to a morphism

$\pi_n(X,x_0) \to \pi_n(Y,y_0)$

These maps are easy to define but...
There is another set of groups $H_n(X)$ (abelian for $n \geq 0$) which is functional. They are harder to define but easier to compute. Before defining them we need some algebraic ideas.
Def A chain complex $C$ is a collection of abelian groups and homomorphisms

$$\begin{align*}
C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} C_2 \xrightarrow{d_2} C_3 \xrightarrow{d_3} C_4 \ldots
\end{align*}$$

with $d_n d_{n+1} = 0$ for all $n \geq 1$.

A term $x \in C_n$ is an $n$-chain

$\ker(d_n) = \{ x \mid d_n(x) = 0 \}$, the $n$-cycles

$\im(d_n) = \{ y \mid y = d_{n+1}(z) \}$, the $n$-boundaries

$x_n$ is called the $n$th
Since $d_n \cdot d_{n+1} = 0$, $B_n \subset Z_n$

Boundary operator

Image of incoming map

Kernel of outgoing map

$H_n(C) = Z_n / B_n$

$n$th homology of $C$