Back to the excision axiom.

Let $U$ be a covering of $X$, i.e., a collection of subsets $U_\alpha$ s.t.
\[ \bigcup \alpha \in \{ U_\alpha \} = X. \]
Let $\delta^m_\alpha (x)$ be the angular chain co generated by $x$ in $\Delta^m$.

Prop. 2.21. There are chain homotopy
equivalence \[ S^2_n(x) \Leftrightarrow S^1_n(x) \]

Proof: Define the subdivision operator

\[ S^1_n(x) \xrightarrow{S^1} S^0_n(x) \]

\[ G \xrightarrow{\text{sum of the } (n+1)!} \text{smallest complex from transcent} \]

\[ \text{subdivision} \]

One can show this gives a CHF from \( S^1_n(x) \) to itself, see pp 121-123.
Let $\Delta^n \to X$ be a map. $\Delta^n$ is covered by the pre-images $\Delta^n(U_\alpha)$. We can replace this by a finite covering since $\Delta^n$ is compact. If we subdivide $\Delta^n$ enough (say $m_0$) times, each little simplex will be small w.r.t. to $U_\alpha$, i.e. each is contained in some $\Delta^n(U_\alpha)$. Let

$$S_m(x) \xrightarrow{D_m} S_{m+1}(X)$$

be a chain homotopy between $1_{S_{m+1}(X)}$. 
and $S^n$.

We want to define $S_n(x) \to S^n_{x}(x)$.

We will do this by constructing a chain with $\Delta$ between $p$ and $1$.

$S_n(x) \xrightarrow{\Delta^n} S^m(x) \xrightarrow{D_m(e)} S^{n+1}(x)$

We can use this map to define

$S_n(x) \to S^n_{x}(x)$

by induction on $n$. 
To prove 2.21 we need to show both

1. \( S^U(X) \to S(X) \to S^U(X) \)

2. \( S(X) \to S^U(X) \to S(X) \)

are chain homotopies to the identity

1. in the identity on \( S^U(X) \).

For 2.7 L is the derived chain homotopy QED

We have proved the excision axiom.

We now derive the Mayer-Vietoris sequence.
We have elements \( A, B, C \) of \( X \), such that:
\[
\text{int}(A) \cup \text{int}(B) = X. \quad \text{MVS in}
\]

\[
\vdots \to H_m(A \cap B) \to H_m(A) \to H_m(B) \to H_m(X) \to H_{m-1}(A \cap B) \to \vdots
\]

Let \( \mathcal{U} = \{A, B\} \)

\[
\mathcal{S}_\mathcal{U}(X) = \mathcal{S}_\mathcal{U}(A + B)
\]

is a chain generated by elements contained in \( A \) or \( B \).

Then we know that \( \mathcal{S}_\mathcal{U}(A+ B) \cong \mathcal{S}_\mathcal{U}(X) \)

by 2.021. We have a SES of
chain $\Rightarrow$

$0 \to S_x(A \cap B) \to S_x(A) \otimes S_x(B) \to S_x(A \cup B) \to 0$

The resulting LES in $H_*$ is the Mayer-Vietoris sequence.

**Big Theorem:** Any functor satisfying the Eilenberg-Steenrod axioms is isomorphic to singular homology. The axioms determine $H_*(X)$ for any $X$. 
Some related questions:

1) Can we describe $H_x(X \times Y)$ in terms of $H_x(X)$ and $H_x(Y)$? This is related to an algebraic question:

Given chain complexes $C'$ and $C''$ describe $H_x(C' \otimes C'')$ in terms of $H_x(C')$ and $H_x(C'')$.

Aside: Why are these related?

Suppose we could answer the algebraic question. There is a
nice map (to be defined later maybe)

\[ S_\times(X) \otimes S_\times(Y) \rightarrow S_\times(X \times Y) \]

It is known to be a CFE. I will not prove it.

You might guess

\[ H_\times (C \otimes C') = H_\times(C) \otimes H_\times(C') \]

NOT TRUE.

2) Given a chain complex \( C \)
and an abelian gp \( A \), consider
\[ H_\times(C \otimes A) \]
which is not \( H_\times(C) \otimes A \)
in general.
A useful case of this is $A = \text{field}$, e.g. $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$. If $C'$ and $C''$ are chain complexes of vector spaces over a field $K$, then

$$H_x (C' \otimes_K C'') = H_x (C') \otimes_K H_x (C'').$$

It is useful to define

$$H_x (X; A)$$

[not to be confused with $H_x (X; A)$ for $A \subset X$]
as $\text{H}_*(S_*(X) \otimes A)$, the homology of $X$ with coefficients in $A$.

If $A$ is a field $k$, then

$$\text{H}_*(X \times Y; k) = \text{H}_*(X; k) \otimes_k \text{H}_*(Y; k).$$

3) Given a chain complex $C$ and an abelian group $A$, consider the cochain complex $\text{Hom}(C, A)$.

$$\cdots \to \text{Hom}(C_{n+1}, A) \to \text{Hom}(C_n, A) \to \text{Hom}(C_{n-1}, A) \to \cdots$$
The bottom row is called a cochain ex. It has cohomology gaps. We define $H^*(X'; A) = H^*(\text{Hom}(S_x(x), A))$, the cohomology of $X$ with coefficients in $A$.

**Why do this?**

\[ \Delta : X \to X \times X \]

Let $A = \text{ring } R$

\[ X \to (x, x) \]

\[ H^*(X; R) \to H^*(X \times X; R) \to H^*(X; R) \otimes H^*(X; R) \]
This map is called cup product. Given \( \alpha \in H^i(X; \mathbb{R}) \) and \( \beta \in H^j(X; \mathbb{R}) \), we get \( \alpha \smile \beta \in H^{i+j}(X; \mathbb{R}) \).

This is a useful structure.

How do we calculate \( H_* \left( \text{Hom}(C, A) \right) \) in terms of \( H_* C \) and \( A \)?

It is met \( \text{Hom}(H_* C, A) \) in general.
This brings us to homological algebra.

The basic result: $\mathfrak{O}$ and $\text{Hom}$ fail to preserve exactness.

Given a short exact sequence

\[ 0 \to A \xrightarrow{i} B \xrightarrow{\rho} C \to 0 \]

and an abelian gp $D$, what about $A \otimes D \to B \otimes D \to C \otimes D$?

E.g. $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2/2} 0$
and let $D = z/2$. We get

$$z/2 \rightarrow D \rightarrow z/2$$

In general, $j \otimes D$ is onto, but $i \otimes D$ may not be 1-1, and $j \otimes D = 0$. Same goes for $\text{Hom}(\cdot, D)$ on $\text{Hom}(D, \cdot)$. Both fail to preserve exactness.