Use MVS to find $H \times K$ where $K = \text{Klein four}\),

$K = M \cup M \cup M \cup M \\
M = \text{Moebius band}$

\[
\begin{array}{c|cc}
0 & 0 & 0 \\
2 & 2 & 2 \\
1 & 1 & 2 \\
0 & 0 & 2 \\
\end{array}
\]
Def: The standard $n$-simplex $\Delta^n$

(a) in $\mathbb{R}^{n+1}$: $\{(x_0, x_1, \ldots, x_n) : x_i = 0, \sum x_i = 1\}$

(b) $\mathbb{R}^n$: $\{(x_1, x_2, \ldots, x_n) : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1\}$

Properties: $\Delta^n \subset \mathbb{R}^n$

- It has $(n+1)$ faces $\times \Delta^{n-1}$
- $n = 2$, $\Delta^2$ has 3 edges
- For each $x \perp x_1 = 0$, $x_1 = x_2$, $x_2 = 1$
In general, we have \( n+1 \) such subsets defined by
\[
\begin{cases}
t_1 = 0, & t_i = t_{i+1}, \text{ for } 1 \leq i \leq n-1, \quad t_n = 1
\end{cases}
\]

In terms of \( (a) \), we can set
\[
x_i = 0 \quad \text{for any } i \text{ with } 0 \leq i \leq n
\]
Thus we get \( (n+1) \)-maps \( \Delta^n_{i+1} \rightarrow \Delta^n_i \)
for \( 0 \leq i \leq n \).

Def. A \( \Delta \)-complex is a union of simplices of various dimensions.
Glued together by identifying certain faces

Examples

(1) $T^2$

(2) $RP^2$

(3) $K$
We will attach a chain complex $C^\Delta(x)$ to a $\Delta$-complex $X$ as follows:

1) $C_i^\Delta(x) = \text{free abelian group with one generator for each } \Delta^i$ in $X$.

In our 3 examples:

$C_0 = \mathbb{Z}^N \oplus \mathbb{Z}^2$, $C_1 = \mathbb{Z}^2 \oplus \mathbb{Z}^2$,
$C_2 = \mathbb{Z} \oplus \mathbb{Z}$.

Let the faces of $\Delta^n$ be $\Delta^i_j$ for $0 \leq i \leq n$. 
\[ \Delta^n \stackrel{i_n}{\rightarrow} \Delta^n \Delta \rightarrow I \]
\[ C_{n+1}(X) \rightarrow d_n \rightarrow C_n(X) \]
\[ \sum_{0 \leq j \leq n} [x, s_j^m] \leftarrow [x]. \text{ The signs are needed to make this a chain.} \]

Lemma: For \( d_n \) defined as above,
\[ d_{n+1} d_n = 0. \]
Computation for \( D \)

\[ C_2(x) = 2 \otimes 2 \] generated by \([U]\) and \([L]\)

\[ C_1(x) = 2 \otimes 2 \otimes 2 \] generated by \([a]\), \([b]\) and \([c]\)

\[ C_0(x) = 2 \] generated \([U]\)

\[ a_1 [a] = [U] - [V] = 0 = a_1 \] \([L]\) = \([C]\)

\[ a_2 [U] = -[a] - \Sigma [b] + [C]\]

\[ d_2 [L] = +[a] + [L] - [C]\]

\[ a_2 (C + L) = 0 \] \( \Rightarrow \) \( H_2 = \mathbb{Z} \)

\[ H_1 = \mathbb{Z} \oplus \mathbb{Z} \]

\[ H_0 = \mathbb{Z} \]
This is the same answer we got with the MVS. The other two examples give the same answers as MVS. This is encouraging.

1. Not all spaces can be constructed in this way.
2. There are many ways to make a suitable space into a D-complex.
What to do??

Throw in the kitchen sink

Let $X$ be a topological space.

$C_m(x) = \text{free abelian $qB$ generated by all continuous maps}$

$\Delta^n \circ \delta X$

$\Delta^n \xrightarrow{i} \Delta^n \xrightarrow{\delta} X$

$d_m[\delta] = \sum (-1)^j [\delta i^m_j]$
The resulting chain complex is denoted by $S_j(X)$, the singular chain complex of $X$.

Good things:

1) $S_j(X)$ depends only on the topology of $X$, and not on any additional structure.

2) A map $X \rightarrow Y$ induces a chain map $S_j(X) \rightarrow S_j(Y)$. 


Bad news

$S_x(x)$ is uncountably generated

in most cases, so

$H_x S_x(x)$ could be hard to compute

Example $X = \text{point}$.

$S_m(x) = \mathbb{Z}$
The map $d_n : S_n(x) \to S_{n-1}(x)$ is multiplication by $\sum v^i = \{ 0 \}$ if even
multiplication by $\sum v^i = \{ 0 \}$ if odd

$H_n S_*(X) = \begin{cases} 2 & \text{for } n = 0 \\ 0 & \text{for } n > 0 \end{cases}$ This is the dimension axiom.

Then let $X$ be a path connected space
Then $H_0(X) = 2$. More generally
$H_0 X$ is the free abelian group generated
by the path connected components of $X$
Proof: \( S_0(X) = \) free abelian group generated by the points of \( X \)

Let \( x_0, x_1 \in X \) and let \( \gamma \) be a path from \( x_0 \) to \( x_1 \). Then \( [x_0], [x_1] \in S_0(X) \) and \( [\gamma] \in S_1(X) \)

\[ d_1([\gamma]) = [x_0] - [x_1] \]

This means each \( x \in X \) gives the same element in \( H_0(X) \).

Hence \( H_0(X) = \mathbb{Z} \) \( \text{QED} \)
Next class on 3/24.