Another example of Poincare duality:

\[ X = \mathbb{R}^n, \quad H_i(x, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = n, \\
\mathbb{Z}/2 & \text{for } 0 < i \leq n, \\
0 & \text{else} \quad \text{n odd} \end{cases} \]

PD from integer coefficients applies when \( n \) is odd and says

\[ H_i(x, \mathbb{Z}) \cong H_{n-i}(x, \mathbb{Z}). \]
For all $n$ we have mod 2 Poincaré duality:

$$\hat{H}_i(X; \mathbb{Z}/2) = \hat{H}^i(X; \mathbb{Z}/2) \cong \bigoplus_{0 \leq i \leq n} \mathbb{Z}/2 \quad \text{for} \quad n \geq 1$$

So $\hat{H}_i(X; \mathbb{Z}/2) \cong \hat{H}^{n-i}(X; \mathbb{Z}/2) \cong \mathbb{Z}/2$ for $0 \leq i \leq n$.

We also know $\chi^*(\mathbb{RP}^n; \mathbb{Z}/2) = 2/2^5 \chi(X; \mathbb{Z}/2)$

where $X \in \hat{H}_1$.

This means there should be a class $Y \in \hat{H}_{n-1}(\mathbb{RP}^n; \mathbb{Z}/2)$ Poincaré dual to $X$ such that self-intersections of $Y$ are dual to powers of $X$. 
For $n=3$, $Y$ is represented by a linear inclusion $\mathbb{R}P^3 \hookrightarrow \mathbb{R}P^3$

\[
[a, b, c] \mapsto 2a [a, b, c, 0] + b^2 [a, b, 0, c] + c^2 [a, 0, b, c]
\]

The intersection of these two is $\mathbb{R}P^1 = \{ [a, b, 0, 0] : a^2 + b^2 = 0 \} \subseteq \mathbb{R}P^3$, representing a class in $H_1$ dual to $x^2$.

For the 3-fold self-intersection, let $[a, b, c] \mapsto [a, 0, b, c]$

The intersection of the 3 images is

\[
\{ [a, 0, 0, 0] : a \neq 0 \} \cong \mathbb{R}[0, 0, 0]
\]
The representation of the element in $H_3$ dual to the generator $X^3 \in H_3$.

This can be generalized to the $n$-dimensional case and to the complex case. Modulo Poincaré Duality, this proves that $H^*(\mathbb{R}P^n; \mathbb{Z}/2)$ and $H^*(\mathbb{C}P^n; \mathbb{Z})$ have the stated ring structures.

New topic: Fiber bundles

Recall a covering $\tilde{\mathcal{X}} \to \mathcal{X}$ is a map

such that each $\tilde{x} \in \tilde{\mathcal{X}}$ has a neighborhood $U$ with

$p^{-1}(U) \cong \mathcal{X} \times D$ where $D$ is discrete.
In a fiber bundle we replace \( D \) by some space \( F \) called the fiber of \( p \). We will denote this by

\[
\begin{align*}
F & \xrightarrow{\pi} E & \xrightarrow{p} B &= \text{base space} \\
\text{fiber} & \xrightarrow{\pi} \mathcal{X} & \xrightarrow{\pi} X &= \text{total space}
\end{align*}
\]

For each \( b \in B \), \( p^{-1}(b) \approx F \).

**Example**

1) \( E = F \times B \) and \( p = \pi_2 \): projection onto second factor. Trivial example.

2) \( F = \) any covering of \( B \), \( F \) indiscrete.

3) The Hopf mapping.
4. Let $G$ be a topological group, e.g., a Lie group. Let $K \subset H \subset G$ be closed subgroups, not necessarily normal.

Then the maps

$$H/K \rightarrow G/K \rightarrow G/H$$

are exact at $H/K$.

This is a fiber sequence, i.e., $p^{-1}(x) = H/K$ for each $x \in G/H_0$. 
Each of the examples in $\mathbb{B}$ has this form.

Def: An $\mathbb{IR}^n$-bundle over a space $\mathbb{B}$ is a fiber bundle with base $\mathbb{B}$ and fiber $\mathbb{IR}^n$, such that given 2

inads $U_1$ and $U_2$ with homeomorphisms

$h_1: \beta^{-1}(U_1) \to U_1 \times \mathbb{IR}^n$ and $h_2: \beta^{-1}(U_2) \to U_2 \times \mathbb{IR}^n$

with $U_1 \cap U_2 = W 
eq \emptyset$, the composite

$W \times \mathbb{IR}^n \xrightarrow{h_2^{-1}} \beta^{-1}(W) \xrightarrow{h_1} W \times \mathbb{IR}^n$

$(w, x) \mapsto (w, \beta_w(x))$
in such that \( f \) is linear on \( \mathbb{R}^n \).

**Example** Let \( M \) be a smooth manifold embedding \( \mathbb{R}^{n+1} \). We get a collection tangent \( n \)-planes, one for each \( x \in M \).

Let \( E = \{ (x, y) \in M \times \mathbb{R}^{n+1} : y \in \text{tangent } n \text{-plane of } x \} \).

There is a map \( E \xrightarrow{p} M \)

\[
(x, y) \mapsto x
\]

\( p^*(x) = \text{tangent } n \text{-plane of } x \subset \mathbb{R}^n \).

Smoothness implies the needed linearity.
A variation: Replace tangent vectors by normal vectors and get an \( \mathbb{R}^N \)-bundle over \( M \).

Let \( x \) and \( y \) be vector bundles over \( X \) of dimensions \( m \) and \( n \) with total spaces \( E \) and \( E' \). Then we have a map \( E \times E' \to X \times X \) which is an \( \mathbb{R}^{m+n} \)-bundle over \( X \times X \). 
null space \rightarrow E_1 \times E_2 \\
\downarrow \\
\times \xrightarrow{\text{diagonal}} \times \times \times \times \\
E = \{(x, y) \in E_1 \times E_2 : \rho_\alpha(x) = \rho_\beta(y)\}

This is an IR^{m+n}-bundle over \Sigma

called the Whitney sum of \alpha and \beta.

HASSLER WHITNEY

X = M^n, \alpha = tangent bundle

\mathbb{R}^{m+k}, B = normal bundle
\( \mathcal{O} \otimes B \cong \text{trivial } R_{n+k} \text{-bundle} / M \)

\[ = M \times R^{n+k} \]

...but in general \( \mathcal{O} \) is not the trivial \( R^n \)-bundle over \( R \) and \( B \) is also nontrivial.