Recall a smooth manifold $M$ of dimension $n$ has a tangent vector bundle $T(M)$. It can be defined without referring to an embedding $M^n \rightarrow \mathbb{R}^{n+k}$. There is also a normal $k$-plane bundle $D(M)$. We see that $V(M) \otimes D(M)$ is the trivial $\mathbb{R}^{n+k}$-bundle $\mathbb{R}^{n+k} \times M$. $V(M)$ depends on the embedding $M \rightarrow \mathbb{R}^n$, then any two embeddings...
have isomorphic normal bundles.

Def: Let $E$ be the total space of $\mathfrak{T}(M)$
so we have a map $E \to \mathcal{D}M$.
A **tangent vector field of $M$** is a
map $M \to E \to \mathcal{D}M$ s.t. $p \circ \xi = \xi M$.

It assigns to each $x \in M$ a tangent
vector $\xi(x) \in \mathcal{D}^xM$ which varies continuously
with $x$.

**Example**: The zero vector field with
$\xi(x) = 0$ for all $x$. 
Question: Does $M$ have a non-zero (at each point $x \in M$) vector field?

Naray Ball Theorem: $S^{2m}$ does not have a non-zero vector field for any $n > 0$. Each $S^{2m-1}$ does have one.

Proof: Odd dimensional case:
$S^{2m-1} = \text{unit sphere in } \mathbb{R}^n$.
For $x \in S^{2m-1}$, let $x(x)$ be a unit vector pointing in the direction of $ix$ (where $i = \sqrt{-1}$).
Consider the map $i: S^{2n-1} -\rightarrow S^{2n}$

$x \mapsto x$

We know it is not homotopic to the identity.

Suppose that $\xi$ is a non-zero tangent vector field on $S^{2n}$.

We can replace $\xi$ by $\frac{\xi}{|\xi|}$, so we have a field of unit vectors...
This allows us to construct a homotopy between $g$ and $1$. $I \times S^{2n} \to S^{2n}$ such a map assigns to each $x \in S^{2n}$ a path from $x$ to $-x$, and these paths vary continuously with $x$.

Use $s$ to determine the direction of each such path. This $h$ does not exist because $g \neq 1_{S^{2n}}$. $s$ does not exist. QED.
Back to fiber bundles

\[ \begin{array}{ccc}
F & \to & E \\
\downarrow & & \downarrow \\
\text{fiber} & \to & \text{total space} \\
\downarrow & & \downarrow \\
B & \to & \text{base space}
\end{array} \]

\[ \pi^{-1}(b) \cong F \quad \text{for any} \ b \in B \]

The map \( p \) has the homotopy lifting property (HLP).

Given \( (x, x_0) \times I \to (E, x_0) \)

the diagram

\[ \begin{array}{ccc}
(X, x_0) \times I & \xrightarrow{b} & (B, x_0) \\
\downarrow & & \downarrow \\
\Sigma X \times I & \xrightarrow{\Sigma b} & \Sigma B \times I
\end{array} \]

\[ \exists \tilde{b} : X \times I \to E \quad \text{s.t.} \quad \tilde{b} \circ \iota = b \quad \text{and} \quad \tilde{b} \circ \partial_0 = \tilde{b} \quad \text{and} \quad \tilde{b} \circ \partial_1 = \tilde{b} \cdot b \]
A fiber bundle always has this.

Def: A map $E \to B$ has the HLP if the above statement is true for
and $f_0$. $E$ is a Serre fibration if it has this for $E = I^n$ for any $n$.

Def: Given a pointed space $(B, b_0)$, its path space $PB$ is the space of
maps $(I, 0) \to (B, b_0)$, i.e., paths starting at $b_0$. There is a map
$PB \to B$ $w \mapsto w(1)$.

This is the path fibration of $B$. 
Facts: 1) \( p^{-1}(x) \) = set of paths from \( x_0 \) to \( x \) \( \subseteq \mathbb{B} \)
   \( p^{-1}(x_0) \) = set of closed paths at \( x_0 \)
   \( = \mathbb{S}^1 \mathbb{B} = \text{loop space of } \mathbb{B} \).

2) \( p \) is always a fibration.

3) \( \mathbb{PB} \) is contractible. You can shrink any path to the constant path.

Prop: Any map \((X,x_0) \to (Y,y_0)\) is homotopy equivalent to a fibration.
Proof. Replace $Y$ by the mapping cylinder $M_b = X \times I \cup Y / (x,d) \sim f(x)$ for each $x \in X$.

$M_b$ is homotopy equivalent to $Y$.

Replace $X$ by $\hat{X} = \text{space of paths in } M_b \text{ starting in } X \times \{0\}$.
There is a map $\tilde{\tilde{\mathcal{W}}} \rightarrow \mathcal{X}$.

It is a little equivalence.

Let $y = M_0 / \text{path from } (x_0, 0) \text{ to } (x_0, 1) = y_0$.

The map $\tilde{\tilde{\mathcal{W}}} \rightarrow \mathcal{X}$.

$\mathcal{W} \rightarrow \mathcal{W}(1)$

Then we have a diagram:

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  \tilde{\tilde{\mathcal{W}}} \rightarrow \mathcal{X} \rightarrow \mathcal{Y}
  \downarrow \quad \quad \downarrow
  \tilde{\tilde{\mathcal{W}}} \rightarrow \mathcal{X} \rightarrow \mathcal{Y}
  \downarrow \quad \quad \downarrow
  \mathcal{W} \rightarrow \mathcal{W}(1) \rightarrow \mathcal{Y}
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Diagram commutes up to $\tilde{\tilde{\mathcal{W}}}$.
Then $F$ is always a fibration. It has a homotopy theoretic fiber $\tilde{F}^{-1}(\tilde{y}_0) = \tilde{F}$. If our original map $X \to Y$ was a fiber bundle with fiber $F$, then $F$ is homotopy equivalent to $\tilde{F}$.

Then for $X \to Y$ and $\tilde{F}$ as above there is a LES of homotopy groups

$$\tilde{\pi}_m(F) \to \tilde{\pi}_m(X) \to \tilde{\pi}_m(Y) \to \tilde{\pi}_{m-1}(F)$$

i.e. given a fiber bundle
Examples

Let $\tilde{X} \to X$ be a covering

with discrete fibers $D$ and $\tilde{X}$ and $X$

Path connected

$\pi_0 \tilde{X} = \text{set of path components}$

$\pi_k \tilde{X} \cong \pi_k X$ for $k \geq 1$

$\pi_k D = 0$
Subexample \( 2 \rightarrow \mathbb{R} \rightarrow S^1 \)

We find that for \( k > 1 \),

\[ \pi_k S^2 = \pi_k \mathbb{R} = 0 \quad \text{unless} \quad \mathbb{R} \text{ is contractible.} \]

2) Let \( S^1 \longrightarrow S^3 \longrightarrow S^2 \) be the Hopf fibration:

\[ \pi_2(S^1) \rightarrow \pi_2(S^3) \rightarrow \pi_2(S^2) \rightarrow \pi_1(S^1) \rightarrow \cdots \]

We know \( \pi_i(S^n) = 0 \) for \( i < n \),

and \( \mathbb{R} \) in \( S^n = \mathbb{R} \)
For \( k \geq 2 \), we have

\[ \pi_k(S^1) \to \pi_k(S^2) \to \pi_k(S^2) \to \pi_{k-1}(S^1) \]

\[ U \quad \pi_0 S^2 = \mathbb{Z} \quad 0 \]

3) Let \( \mathbb{R}^n \to E \to X \) be a vector bundle.

\[ \pi_k(\mathbb{R}^n) \to \pi_k E \to \pi_k X \to \pi_{k-1}(\mathbb{R}^n) \]

\[ U \quad \mathbb{Z} \quad 0 \]

4) The path fibration

\[ \text{Fix} X \to \text{P}_{S^1} X \to X \]
\[ \mathcal{P}X \rightarrow \mathcal{P}X \xrightarrow{\mathcal{G}} \mathcal{T}_{\mathcal{K}X} \mathcal{S}2X \rightarrow \mathcal{T}_{\mathcal{K}X+1} \mathcal{R}N \]

\[ \mathcal{T}_{\mathcal{K}X} \mathcal{S}2X = \mathcal{T}_{\mathcal{K}X+1} \mathcal{X} \]