There are 5 Platonic solids.

<table>
<thead>
<tr>
<th>Solid</th>
<th>V</th>
<th>E</th>
<th>F</th>
<th>( \chi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Cube</td>
<td>8</td>
<td>12</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>Octahedron</td>
<td>6</td>
<td>12</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>Dodecahedron</td>
<td>20</td>
<td>30</td>
<td>12</td>
<td>2</td>
</tr>
<tr>
<td>Icosahedron</td>
<td>20</td>
<td>30</td>
<td>20</td>
<td>2</td>
</tr>
</tbody>
</table>

Def Let \( X \) be a finite CW-complex with \( C_i \) cells in dimension \( i \). Then its Euler characteristic \( \chi(X) = \sum (-1)^i \cdot C_i \).
The $X(S)$ is a topological invariant and is equal to $\Sigma S^i h_i$.

where $h_i$ = rank of $H_i(X)$ / torsion = # of summands $\cong$.

Proof: In the cellular chain $C(X)$ of $X$,

let $C_i$ = rank of $C_i(x)$ = # of $i$-cells,

$Z_i$ = rank of $Z_i \subset C_i$ = the gp of cycles,

$B_i$ = $\ker$ $d_i \subset C_i$ = the gp of boundaries,

Then $h_i = Z_i - B_i$ since $H_i = Z_i / B_i$.

$C_i = Z_i + B_{i-1}$ since there is a SES

$0 \rightarrow Z_i \rightarrow C_i \rightarrow B_{i-1} \rightarrow 0$. 

\[
\Sigma E_i^a C_i = \sum_{i=1}^n (E_i^a \cdot (3_{i} + b_{i-1})
\]
\[
= \sum E_i^a z_i + \sum E_i^a b_{i-1}
\]
\[
= \sum E_i^a (z_i - b_i)
\]
\[
= \sum E_i^a h_i \quad \text{QED}
\]

Are there any other Platonic solids?
Suppose we have such with
- \(V\) vertices,
- \(E\) edges,
- \(F\) faces,
- \(p\) edges on each face and
q edges on each vertex.

Then

\[ F = 2E \text{ since each edge belongs to 2 faces} \]

\[ qV = 2E \text{ since each edge has 2 vertices} \]

\[ F = \frac{2}{q}E \quad V = \frac{2}{q}E \quad p, q \geq 3 \]

\[ \chi = V - E + F = E \left( \frac{2 - 1 + \frac{1}{q}}{q} \right) = 2 \]

We need \( \frac{2}{q} + \frac{1}{q} > 1 \). The only possible values for \((p, q)\) are \((3, 3), (3, 4), (4, 3), (5, 5)\) and \((5, 3)\)
These correspond to the 5 Platonic solids.

Suppose we replace \( S^2 \) by the torus \( T^2 \), with \( x(T^2) = 0 \).

Then we find \( \frac{2}{\theta} - 1 + \frac{2}{\theta} = 0 \).

Then \( (\phi, \eta) = (3, 6), (6, 3) \) or \( (4, 4) \).

The equation above gives no info about \( E \) and hence about \( U \) and \( F \).

Recall that there is a covering:
\[ \mathbb{R} \to S^1 \]
\[ \mathbb{R}^2 \to S^1 \times S^1 = T^2 \]
This is the universal covering of $T^2$.
In the plane we get three possible pictures.

$(3,6)$

$(4,4)$

$(6,3)$

To get to the torus, choose an action of the group $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{R}^2$ that leaves the pattern invariant.
Examples: For $(p,q) = (3,6)$ let $G$ be the gp generated by $\alpha$ and $\beta$ as shown above. The orbit space is a torus with $F = 18$, $E = 27$, $V = 9$, $V - E + F = 0$. $\alpha$ and $\beta$ could be replaced by any 2 linearly independent vertices.

Similar remarks apply to grid and honeycomb.
Consider a surface with genus \( \geq 1 \).

\[ V = 5 \quad E = 10 \quad F = 5 \]
\[ V - E + F = 0 \]

The surface of the model
\[ V = 12 \quad E = 30 \quad F = 12 \]

\[ \gamma = \text{rank of } H_1 \]
\[ \chi = 2 - 2g \]

\( g \) = genus 3
\[ p = 5 \quad g = 5 \]
\[ \chi = V - E + F = 12 - 30 + 12 = -6 \]
\[ g \geq 4 \]

We have a surface of genus 4 covered by 12 pentagons with 5 meeting at each vertex.

Final: 5/6 2 PM