Recall: For a chain $C$ of free modules over a PID $R$ (e.g., $\mathbb{Z}$) and an $R$-module $A$,

$$N_n(C \otimes A) = N_n(C) \otimes A \oplus Tor_1^R(N_{n-1}(C), A)$$

$$N_n(C', A) = \text{Hom}(N_n(C), A) \oplus \text{Ext}^1_R(N_{n-1}(C), A)$$

**Notation:** When working over a PID, it is common to write $\text{Tor}$ for $Tor_1$ and $\text{Ext}$ for $\text{Ext}^1$. 
Example: Let $C$ be $Z^{u} Z^{v} Z^{2}$

$C_0, C_1, C_2$

$H_i C = \begin{cases} 
2 & \text{for } i = 0 \\
2/2 & \text{for } i = 1 \\
0 & \text{for } i \geq 2
\end{cases}$

$A = 2/2$ so $C \otimes A$ is $2/2 e^{0} 2/2 e^{0} 2/2$

$H_i (C \otimes A) = \begin{cases} 
2/2 & \text{for } 0 \leq i \leq 2 \\
0 & \text{for } i > 2
\end{cases}$

$H_i (C) \otimes A = \begin{cases} 
2/2 & \text{for } i = 0, 1 \\
0 & \text{for } i \geq 2
\end{cases}$
\[ \text{Tor}_i(\mathbb{H}_0(C), \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{for } i = 1 \\ 0 & \text{for } i \neq 1 \end{cases} \]

Similarly we find
\[ \tilde{H}^i(\mathbb{C}, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{for } 0 \leq i \leq 2 \\ 0 & \text{for } i > 2 \end{cases} \]

Let \( A = \mathbb{Z} \). Then \( \text{Hom}(\mathbb{C}, \mathbb{Z}) \) is the cochain complex

\[
\begin{array}{ccc}
& \mathbb{Z} & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z} \\
\oplus & 0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z} \\
\end{array}
\]

\[ \text{Hom}(\mathbb{Z}/2, \mathbb{Z}) = 0 \] and \( \text{Ext}^1(\mathbb{Z}/2, \mathbb{Z}) = \mathbb{Z}/2 \)
The UCT gives the right answer.

Thm 3.5.5 (Künneth Theorem). Let $C$ and $C'$ be chain complexes of free abelian groups. Then there is a split SES

$$0 \to \bigoplus_{0 \leq i \leq n} H_i(C) \otimes H_{n-i}(C') \xrightarrow{\partial} H_n(C \otimes C') \xrightarrow{\partial} \bigoplus_{0 \leq i \leq n-1} \text{TCA}_i(H_i(C) \otimes H_{n-i-1}(C')) \to 0$$

Proof: More of the same.
Example let $C' = C$ - chain $C$ of previous example
let $a_i$ be generator of $C_i$ for $0 \leq i \leq 2$
and let $B = COC$. Then

$B_0 = \bigoplus \{a_0 \otimes a_0^2\}$  
$B_1 = \bigoplus \{a_1 \otimes a_0, a_0 \otimes a_1\}$
$B_2 = \bigoplus \{a_2 \otimes a_0, a_1 \otimes a_1, a_0 \otimes a_2\}$  
$B_3 = \bigoplus \{a_2 \otimes a_1, a_1 \otimes a_2\}$  
$B_4 = \bigoplus \{a_2 \otimes a_2\}$

$a_2 \rightarrow 2a_1$, $a_1 \rightarrow 0$
$a_0 \rightarrow 0$
$a_2 \otimes a_0 \rightarrow 2a_1 \otimes a_0$
$a_0 \otimes a_2 \rightarrow 2a_0 \otimes a_1$
$a_2 \otimes a_1 \rightarrow 2a_1 \otimes a_1$
$a_1 \otimes a_2 \rightarrow -2a_1 \otimes a_1$
$a_2 \otimes a_2 \rightarrow 2a_0 \otimes a_2 + 2a_2 \otimes a_1$
\[ H_m(C \otimes C) = 2(\alpha \alpha \alpha_2 + \alpha_2 \alpha_\alpha_1) \]

<table>
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<th>( n )</th>
<th>( \alpha ) term</th>
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<td>( \alpha \alpha \alpha \alpha_1 )</td>
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Back to topology

For a space \( X \) and an abelian group \( A \)
It is useful to consider the spaces
\[ H_x^*(X, A) := H_*(S_x(x) \otimes A) \]
\[ H^*(X', A) := H^* \operatorname{Hom}(\mathcal{S}_x(x), A) \]
We can describe these in terms of \( H_*(X) \) and \( A \).

Given spaces \( X \) and \( Y \), we want to describe \( H_*(X \times Y) \) in terms of \( H_*(X) \) and \( H_*(Y) \).
Topological Künneth Theorem

\[ H_n(X \times Y) = \bigoplus_{0 \leq i \leq n} H_i(X) \otimes H_{n-i}(Y) \]

This would follow from the algebraic Künneth theorem if we knew that

\[ S_x(X \times Y) \text{ is CHF to } S_x(X) \otimes S_x(Y) \]
This is true but the proof is very tedious. BAD.

We can avoid this awful proof by restricting our attention to CW complexes.

OUTLINE

1) A CW complex is defined in terms of cells in various dimensions and attaching maps, as explained before.

2) For $C_\star(X)$, the cellular chain $C_\star$
of $X$ by

$$C_n(X) = \text{free abelian gp generated by the n-cells of } X$$

The boundary map is defined in terms of attaching maps.

Thus $H_n(C_n(X)) = H_n(X)$ as previously defined.

Good thing $C_n(X)$ is much smaller than $S_n(X)$. In many cases $C_n(X)$ is finitely generated.
3. Given CW cxs $X$ and $Y$, we can define a CW-structure on $X \times Y$ so that

$$C_\ast(X \times Y) \cong C_\ast(X) \otimes C_\ast(Y)$$

This means we can apply 3B5 directly to get $\pi_\ast(X \times Y)$.

Good things about CW cxs:

- They are very versatile
- $\Delta$-complexes are CW-cxs
where each cell is a simplex.

Any compact manifold or algebraic variety (over \( \mathbb{R} \text{ or } \mathbb{C} \)) in a CW-complex. Homotopy theorists like to study mapping spaces

\[
\text{Maps}(X, Y) = \text{set of cont. maps } X \to Y \text{ with compact-open topology.}
\]

This is not usually a CW-complex.
but if $X$ and $Y$ are CW-complexes, $\text{Map}(X,Y)$ is homotopy equivalent to a CW-complex. This is a theorem due to Milnor. Moral: CW-complexes are good enough for most purposes.