Final 2-5 PM here (?!) Tuesday

Two previous finals on course web page.

Review

**Def:** Two maps \( f, g : X \to Y \) are **homotopic** if there is a map

\[ h : I \times X \to Y \] such that \( f_t(x) = h(t, x) \) for

\[ t = 0, 1. \]

**Def:** Two spaces \( X, Y \) are **homotopy equivalent** if there are maps

\[ X \xrightarrow{f} Y \] and \( Y \xrightarrow{g} X \)

such that \( fg \) and \( gf \) are each homotopic to the identity.
Let $(X, x_0)$ be a space with basepoint $x_0$. Consider maps $(I, x_0) \to (X, x_0)$, i.e., closed paths at $x_0$. $\Pi_1(X, x_0)$ is the set of homotopy classes of such maps under basepoint homotopy.

Given by $(f \ast g)(t) = \begin{cases} f(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}

This defines a group structure on the set $[\Pi_1(X, x_0), (X, x_0)]$.

This group is formalized in $X$, i.e.
Given a map \((X, x_0) \to (Y, y_0)\) we get a homomorphism
\[
\pi_1(X, x_0) \xrightarrow{\ell_*} \pi_1(Y, y_0)
\]
if \(X\) and \(Y\) are path-connected, then
\[
\pi_1(X, x_0) = \pi_1(Y, y_0)
\]
(assume \(X\) and \(Y\) are path-connected).

One can also define \(\pi_n(X, x_0)\) in terms of maps \(\mathbb{S}^n \to (X, x_0)\)
\[
\pi_n(X, x_0) \text{ is abelian for } n \geq 1.
\]
$\dim \vartheta_1 (S^n, x_0) \cong \mathbb{Z}$

\[ \vartheta_1 (S^n, x_0) = \begin{cases} 0 & \text{for } i \neq n \\ 2 & \text{for } i = n \end{cases} \]

Applications

1) Fundamental Theorem of Algebra
   Every polynomial \( P(x) \) has a root.

2) Brouwer Fixed Point Theorem
   Any map \( D^2 \to D^2 \) has a fixed point, i.e., \( \exists x \in D \) with \( f(x) = x \).
   [Similarly for \( D^{n+1} \)]

3) Borsuk-Ulam Theorem
Given $S^2 \to \mathbb{R}^2$ \[ S^n \to \mathbb{R}^n \] \[ \forall x \in S^2 \text{ with } f(-x) = f(x) \].

Van Kampen Theorem.

Let $X = A \cup B$ with $A \cap B$ and $A \cap B$
path connected. Describe
\[ \pi_1(X) \text{ in terms of } \pi_1(A), \pi_1(A \cap B), \pi_1(B) \]

[There is no analogous theorem for $\pi_2$.]

From this we deduce
\[ \pi_1(\mathbb{R}P^2) = 2/\mathbb{Z} \]
\[ \pi_1(\text{Klein bottle}) = \mathbb{Z}/2 \times \mathbb{Z}/2 \]
\[ \pi_1(\mathbb{R}P^n) = \pi_1(\mathbb{R}P^2) \text{ for } n > 2. \]

Prop: Suppose \( Y \) is obtained from \( X \) by attaching an \( n \)-cell for \( n > 2 \).

Then \( \pi_1(Y) = \pi_1(X) \).

Proof: \( Y \) is a pushout.

\( \begin{array}{ccc}
S^m & \xrightarrow{i} & D^n \\
\downarrow & & \downarrow \\
x & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
\mathbb{R} & \xrightarrow{j} & \mathbb{R}, Y \\
\end{array} \)

\( \pi_1(Y) = \pi_1(X) \) by Van Kampen's diagram.
Covering Spaces

Def: $X \rightarrow X$ is a covering if each $x \in X$ has a neighborhood $U \times U$ with $\bar{\varphi(U)} \times D$ for $D$ discrete.

Such a net is evenly covered by $p$.

Thm: For a covering as above, $\pi_1(\bar{\varphi})$ is always 1-1. (Assume $X$ is path connected and $X \neq \emptyset$.)

In favorable cases, there is a 1-1 correspondence between
Path connected coverings of $X$ and subspaces of $\pi_1 X$.

Conditions on $X$ needed for this

1) Path connected

2) Locally path connected
   (every nbhd of every $x \in X$ has a shrunk nbhd $U'$ which is path conn.)

3) Semi-locally simply connected
   (every nbhd $U$ of every $x \in X$ has an nbhd $U'$ s.t. $\pi_1(U') \to \pi_1(x)$ is trivial.)
For each $X$, there is a covering $\tilde{X}$ with $\pi_1(\tilde{X}) = 0$. The group $\pi_1(X)$ acts freely on $\tilde{X}$ with orbit space $\bar{X} = \tilde{X}/\pi_1(X)$. For any subgroup $H \subseteq \pi_1(X)$, $\pi_1(\bar{X}/H) = H$.

On to homology

Def. A chain complex $C$ is a diagram of abelian groups

\[ 0 \to C_0 \xrightarrow{d_1} C_1 \xrightarrow{d_2} C_2 \xrightarrow{d_3} C_3 \to \cdots \]

s.t. $d_n \circ d_{n+1} = 0$ for each $n \geq 0$. 
Then $H_n(C) = \ker d_n / \text{im } d_{n+1}$.

Idea: Associate a chain $S(X)$ to each topological space $X$ and define $H_n(X) := H_n(S(X))$.

Homology gaps are harder to define but easier to compute than homotopy groups.

Eilenberg–Steenrod Axioms for Homology:
- Homology assigns a graded abelian group $H_n(X, A)$ to each topological space $X$.
A \subset X$, $[A]$ would be empty.
\[ \text{hom}(X, \emptyset) = 0 \text{ for } H^4(X) \]

This is a function.

\[ \text{topological} \quad \mathcal{F} \quad \text{graded} \quad \mathcal{G} \]

\[ \{ \text{pairs} \} \quad \longrightarrow \quad \{ \text{abelian} \} \quad \{ \text{groups} \} \]

i.e. a map \[ (X, A) \longrightarrow (Y, B) \]
induces a hom \[ H_4(X, A) \overset{\phi}{\longrightarrow} H_4(Y, B) \]

Furthermore, there is a natural transformation \[ H_n(X, A) \longrightarrow H_{n+1}(A, \partial) \]
\[ H_{n+1}([A], \partial) \]
Axioms

1) Exactness: There is a LES

\[ \cdots \rightarrow H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \overset{\partial}{\rightarrow} H_{n-1}(A) \rightarrow \cdots \]

\[ (A, \phi) \rightarrow (X, \phi) \rightarrow (X, A) \]

2) Excision: Given \( B \subset A \subset X \) with closure \( \overline{B} \subset \text{int}(A) \), the map

\[ H_n(X - B, A - B) \rightarrow H_n(X, A) \]

is an isomorphism
3) Homotopy: Two homotopic maps \((X, A) \rightarrow (Y, B)\) induce the same hom in \(H_n\).

4) Dimension

\[ H_n\text{(point, } \phi) = \begin{cases} 
2 & \text{for } n = 0 \\
0 & \text{for } n \neq 0
\end{cases} \]

A consequence of the axioms

(Mayer-Vietoris sequence)

Let \( X = A \cup B \) where

\[ \text{closure } (A \cap B) \subset \text{interior } A \]

\[ \text{interior } B \]
There is a LES

\[ \cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots \]

This is a very useful tool for computing \( H_n(X) \).

To define \( H_n(X) \) we need the standard \( n \)-simplex

\[ \Delta^n = \{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0 \leq x_1 \leq \cdots \leq x_n \} \]

\[ \Delta^n = \{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} : t_0 \geq 0 \text{ and } \sum t_i = 1 \} \]

There are \( (n+1) \) inclusions \( \Delta^n \to \Delta^{n+1} \).
$(t_0, \ldots, t_{n-1}) \rightarrow (t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$

for $0 \leq i \leq n$

This enables us to define the singular chain $\sigma X \subseteq X$ of $X$

where $S_m(X) =$ free abelian $g$-graded by all constant maps $\Delta^n \rightarrow X$

$d_m(\sigma) = \sum_{i=0}^{m} (-1)^i \sigma f_i$

$\in S_{m-1}(X)$