Recall we want to understand the cellular chain complex for $\mathbb{RP}^n$. It has the form

$\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \cdots \to \mathbb{Z}$

Claim: $d_k = 1 + (1)^k = \begin{cases} 0 & \text{for } k \text{ odd} \\ 2 & \text{for } k \text{ even} \end{cases}$

$\mathbb{RP}^n$ is the orbit space of a free $G$-action on $S^n$, where $G_1 = G_2$. $S^n$ has a cellular structure with
2 cells in each dimension that are interchanged by $i$. This leads to a cellular chain $\mathcal{C}$ of $R$-modules where $R = \mathbb{Z}[G] = \mathbb{Z}[G]/(\delta^2 - 1)$. It is given by:

$$
\begin{array}{ccccccc}
  & 1 \delta & 1 + \delta & R \rightarrow R \rightarrow R \rightarrow R \rightarrow R \rightarrow R \\
 0 & 1 & 2 & n
\end{array}
$$

The only boundary map giving $H_*(C) = H_*(S^n)$ is $d_1 = 1 + (\delta - 1) \delta$

To pass to the orbit space, apply the function $\mathbb{Z} \rightarrow \mathbb{Z}_+$.
where \( z_+ = R/(1-8) = 2 \) with trivial sb action. This gives \( d_r = 1 + (-1)^r \) as claimed above. This gives the value of \( H+\mathbb{RP}^n \) stated before.

A variation \( G = C_m \) acts freely on \( S^{m-1} \subset C^n \) via multiplication by \( w = e^{2\pi i/m} \). The case \( n = 2 \) is relevant to HW problem #8 due 4/11.
The orbit space $S^{2n+1}/G$, is called a lens space. We will describe a cellular structure on $S^{2n+1}$ with $m$ cells in each dimension, which are cyclically permuted by $G$.

The 0-cells are the points

$$(w^0, 0, 0, \ldots) \text{ for } 0 \leq j < m$$

The 1-cells are centered at

$$(w^{i+1/2}, 0, 0, \ldots).$$

Each 2-cell is attached to the
1-skeleton (\(\approx S^4\)) by a homeo. The 2-cells are centered at 
\((0, w, 0, 0, \ldots)\) 
The 3-cells are centered at 
\((0, w/2, 0, 0, \ldots)\) 
The 4-cells are centered at 
\((0, 0, w, 0, 0, \ldots)\) and so on. 
The cellular chain complex is 
\[ R^1 \rightarrow R^2 \rightarrow R^3 \rightarrow \cdots \] 
\[ 0 \rightarrow H_0 \rightarrow H_1 \rightarrow H_2 \rightarrow \cdots \]
\[ R = \mathbb{Z}[x] = \mathbb{Z}[x] / (x^m - 1) \]
\[ x^m - 1 = (x-1)(1 + x + \cdots + x^{m-1}) \]
\[ d_k = \begin{cases} 1 & \text{if } k \text{ odd} \\ 0 & \text{if } k \text{ even} \end{cases} \]

We can pass to the quotient space by setting \( x \to 1 \), i.e., tensoring with \( \mathbb{Z} = R / (1-x) \). The resulting CK is

\[ \begin{array}{cccccc}
Z & Z & Z & \cdots & Z \\
0 & 1 & 2 & 3 & 2m-1
\end{array} \]
\[ H_i(L^{2n-1}) = \begin{cases} 2 & \text{for } i = 0 \\
2/m & \text{for } i \text{ odd and } 0 < i < 2n-1 \\
\mathbb{Z} & \text{for } i = 2n-1 \\
0 & \text{else} \end{cases} \]

Cartesian products. There is a split

\[
\begin{array}{c}
O \\ 0 \\ 0 \end{array} \xrightarrow{i} \begin{array}{c}
H_i(X) \otimes H_0(Y) \\ 0 \\ 0 \end{array} \xrightarrow{n_1} \begin{array}{c}
H_n(X \times Y) \\ \cdots \\ H_0(X \times Y) \end{array} \rightarrow \\
O \\ 0 \\ 0 \end{array}
\]

Let \( X = Y \).
$H_i^*(X) \otimes H_j^*(X) \to H_{i+j}^*(X \times X) \xrightarrow{\Delta^*} H_{i+j}^*(X)$

Formal properties of cup product

1) Natural in $X$, i.e., given $X \to Y$, we get $H^* X \xrightarrow{\text{cup}} H^* Y$

The diagram commutes.
2) It is associative and distributive, i.e.,
\[(X + Y)Z = XZ + YZ \in H_1^0(X)\]
where \(X, Y \in H_0^1(X)\) and \(Z \in H_0^0(X)\)

\[(X, Y)_2 = X(YZ) \in H_1^{0+1} + H_0^1 Y\]
for \(X \in H_1^0\), \(Y \in H_0^0\) and \(Z \in H_1^0\).

3) There is a unit element in \(H_0^1(X)\) given by \(\mathbb{I}: X \mapsto pt\),
\[H_0^0(X) \leftrightarrow H_0^1(pt) \ni 1 \mapsto 1 \in 1\]

Alternative description: \(H_0\)
in the cochain with value 1 on each 0-simplex (i.e. each point) of $X$. 1. $\alpha = \alpha$ for any $\alpha \in H^n X$.

4) It is commutative up to sign, i.e.
for $\alpha \in H^n X$ and $\beta \in H^n X$,

$$B \alpha = (-1)^{i j} \alpha \beta = \begin{cases} \alpha \beta & \text{if } i \text{ and } j \text{ are odd} \\ \alpha \beta & \text{otherwise} \end{cases}$$

**Examples**

1) $X = \mathbb{R}P^n$,

$$H^i(X;2) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ \mathbb{Z}/2 & \text{for } i \text{ even and } 0 < i < n \\ \mathbb{Z} & \text{for } i = n \text{ and } \text{n is odd} \end{cases}$$
Let $x_{2i}$ denote the generator of $H^{2i}$ for $0 < i < n/2$. Then $x_{2i} = x_i$

$H^i(X; Z_2) = \begin{cases} Z/2 & \text{for } 0 \leq i \leq n \\ 0 & \text{otherwise} \end{cases}$

Let $x_i$ be the generator of $H^i(\text{RP}^n; Z/2)$.

Then $x_i \in X_i$ for $0 \leq i \leq n$.

$H^X(\text{IRP}^n; Z/2) = 2/2 \sum x_i \cap /x_i^{m+1}$

= truncated polynomial algebra
2) \( X = \mathbb{C}P^n \). \( H^i(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z} \) for \( i \) even and \( 0 \leq i \leq 2n \). Let \( x_i \in H^2 \) be a generator. Then \( x_i \) is a generator of \( H^{2i} \).

\[ H^2(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[x] / (x^{n+1}) \]

Application. Consider the spaces \( \mathbb{C}P^2 \) and \( S^3 \vee S^4 \). Both have \( H^2 = \{ \mathbb{Z} \) for \( i = 0, 2, 4 \) \} \). Their cup product structure are
not the same. The cup product in $H^4(S^2 \times S^4)$ is trivial. Let $x \in H^2$ and $y \in H^4$ be generators. Claim $x^2 = 0$.

\[ S^2 \text{ inclusion} \quad S^2 \times S^4 \]
\[ \text{push} \quad S^4 \]
\[ H^*(S^2) \rightarrow H^*(S^2 \times S^4) \]

\[ x \in H^2 \quad \quad \quad \quad \quad x \in H^2 \]

\[ 0 = x^2 \in H^4 = 0 \]
\[ x^2 \quad \quad \quad \quad \quad \quad x^2 = 0 \]
This means $S^2 \vee S^4$ and $\mathbb{C}P^2$ are not homotopy equivalent.

Suppose we have a CW-complex of the form $S^2 \vee E^4$.

Both $S^2 \vee S^4$ and $\mathbb{C}P^2$ fit this description.

In each case, there is an attaching map $2D^4 \to S^3 \to S^2$.

For $S^2 \vee S^4$, it is the constant.

For $\mathbb{C}P^2$, it is the Hopf map $S^3 \to S^2$. 
In general one can ask $x = s^2 u \in \mu^4$ for generators $x \in \mu^2$ and $y \in \mu^4$.

$x^2 = ky$ for some integer $k$.

Any value is possible. Consider the composite $C^2 \subseteq \mathbb{C}^3 \xrightarrow{k} \mathbb{C}^3 \xrightarrow{H_{0,0}} \mathbb{C}^2$

$(z_1, z_2) \rightarrow (z_1^k, z_2) \quad \overbrace{\rightarrow \quad |z_1|^{2/k} + |z_2|^2}^{H_{0,0}}$
Claim the map \([f] \mapsto \text{mult by } k \text{ in } H^3(\mathbb{S}^3)\).

Def. The Hopf invariant of \(S^3 \xrightarrow{f} S^2\) is the integer \(k\) above.

More formally, let \(S^{m-1} \xrightarrow{f} S^m\) and \(X = S^m \cup \mathbb{C}^m\). Let \(x \in H^m(\mathbb{X})\) and \(y \in H^m(\mathbb{Y})\) be generators and \(x^2 = k\).

\(k\) is the Hopf invariant of \(f\).
For $m$ odd, $x^2 = -x^2$

For $m = 1$ we need not have

$H^k X = 2$

$k 
eq 0$

\[
\begin{array}{c}
\Sigma & \cdots & \Sigma \\
\downarrow & \cdots & \downarrow \\
X & \rightarrow & X
\end{array}
\]

$H^1 (X) = \sum \mathbb{Z}$

for $i = 0$

$\mathbb{Z} / k$ for $i = \frac{k}{2}$

$0$ for $i = \frac{k}{2} - 1$

For odd $m > 1$, $k = 0$, let make even. Rephrase it.
$S^{4m-1} \to S^{2m} \text{ for } m > 0$

$x = S^{2m} \cup e^{4m}$

We get a Hopf invariant. What values can it have?

Consider the space $S^{2m} \times S^{2m}$

$S^{4m-1} \to S^{2m} \vee S^{2m} \to S^{2m} \times S^{2m} \xrightarrow{H} S^{2m} \cup e^{4m}$
Exercise: Show f has Hopf invariant 2.