Recall we want compute $\pi_{n+k} X$ where $X$ is $(n-1)$-connected and $k < n$. How to let $n \to \infty$? The category of spectra.

**Def.** A pre-spectrum $X$ is a collection of spaces $X_n$ and maps $X_n \to X_{n+1}$ for $n > 0$.

**Examples:**

1. $X_n = S^n$ and $i_n$ is an equivalence.

2. $X_n = K_n = K(C_{2/3}, n)$; $i_n$ is not an equivalence, but $i_{n+1}$ is.
We define $H_k(X) = \lim_{n \to \infty} H_{m+k}(X_m)$

which induces a homomorphism $H_{m+k}(X_m) \to H_{m+l+k}(X_{m+l})$

$\Pi_k(X) = \lim_{n \to \infty} \Pi_{m+k}(X_m)$

1. $H_k(S^0) = \begin{cases} 2/2 & \text{for} \ k = 0 \\ 0 & \text{for} \ k \neq 0 \end{cases}$
2. $\Pi_k(S^0) = \begin{cases} 2/2 & \text{for} \ k = 0 \\ 0 & \text{for} \ k \neq 0 \end{cases}$

$\Pi_k(S^0) = \text{mystery}$

$H^*(\mathbb{1}/2) = A$ as a graded vector space
Pre def: A map \( X \to Y \) is a collection
\[
\begin{align*}
\Sigma X_n & \xrightarrow{e_n} \Sigma Y_n \\
X_{n+1} & \xrightarrow{f_{n+1}} Y_{n+1}
\end{align*}
\]
for \( n \gg 0 \).

(Too restrictive)

How to get from a prespectrum \( \tilde{X} \) to
a spectrum \( X \): We have
\[
\begin{align*}
\tilde{X}_n & \xrightarrow{i_n} S^1 \tilde{X}_{n+1} & \text{for } n \gg 0 \\
\tilde{X}_n & \xrightarrow{i_n} S^1 \tilde{X}_{n+1} & S^1 \tilde{X}_{n+1} \xrightarrow{i_{n+1}} S^1 \tilde{X}_{n+2} & \text{for } n \gg 0
\end{align*}
\]
Let \( \lim_{i \to \infty} S^i X_{m+1} \). Then \( X_n \) is homeomorphic to \( S^k X_{m+1} \). We can define \( X_{n-k} = S^k X_n \), so \( X_n \) is defined for all \( n \in \mathbb{Z} \).

We can define \( H^*_n (X) \), \( H^*_n (\underline{X}) \) and \( \Omega^*_n (X) \) as before, and the proof of a map is the right one.

Def. A spectrum \( X \) is connective if \( \Omega^*_n (X) = 0 \) for \( i < 0 \).
Example of a noncommutative spectrum

\[ U(n) = n \text{ th unitary group} \]

= \text{gp of } n \times n \text{ unitary matrices} / \mathbb{C}.

\[ U(n) \rightarrow U(n+1) \]

\[ U = \lim_{n \rightarrow \infty} U(n) = \text{stable unitary group} \]

Bott Periodicity Theorem

\[ S^2 U \cong U \quad \text{and} \quad T_i U = \begin{cases} \mathbb{Z} & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases} \]

\[ \mathbb{Z} \cong S^0 \quad \text{and} \quad T_i S^0 = \begin{cases} \mathbb{Z} & \text{if } i \equiv 3 \text{ or } 7 \text{ mod } 8 \\ 0 & \text{otherwise} \end{cases} \]
The unitary group spectrum $X = U$

\[
X_n = \begin{cases} 
    U & \text{if } n \text{ is even} \\
    SU & \text{if } n \text{ is odd}
\end{cases}
\]

$X_0 \to SU \xrightarrow{i} SU^2 X_1 \xrightarrow{i} SU^3 X_2 \xrightarrow{i} \ldots$

$U \xrightarrow{i} SU \xrightarrow{i} SU^2 U \xrightarrow{i} SU^3 U \xrightarrow{i} SU^4 U \xrightarrow{i} \ldots$

$i: U = \begin{cases} 
    2 & \text{if } i \text{ is odd} \\
    0 & \text{if } i \text{ is even}
\end{cases}$ for all $i \in \mathbb{Z}$

Properties of spectrum

We can decompose a spectrum $X$
$Y = \sum_{i=1}^{\infty} X_i$ is defined by $Y_n = X_{n-1}$
$W = \sum X_i$

Then sequences and cofiber sequences are the same thing:

$\tilde{\Pi}_m (X) = \tilde{\Pi}_m (X_0)$ for $n \geq 0$

$\tilde{\Pi}_m (X) = \tilde{\Pi}_{m+k} (X_m)$
Back to the Adams resolution

\[ X = X_0 \xrightarrow{g_0} X_1 \xrightarrow{g_1} X_2 \xrightarrow{} \ldots \]

\[ \begin{array}{ccc}
L_0 & L_1 & L_2 \\
L_0' & L_1' & L_2' \\
\end{array} \]

1) \( L_0 \) is a wedge of suspensions of \( H/2 \)

This means \( H^*(L_0, \mathbb{Z}/2) \) is a free \( A \)-module and \( \Pi_*(L_0) = \text{Hom}_A(H^*(L_0, \mathbb{Z}/2), \mathbb{Z}/2) \) as graded ab.

2) \( H^*(L_0) \) is onto

3) \( X_{0+1} \) is the fiber of \( f_0 \)
This leads to a LES of \( A \)-modules
\[ 0 \to H^*X \xrightarrow{f} H^*L \to H^*\Sigma^1 L \to \cdots \]

This is a free (hence projective) \( A \)-resolution
of \( H^*X \).

Recollections from homological algebra:
Let \( M \) and \( N \) be \( R \)-modules. \( M \) has
a projective (or free) resolution
\[ 0 \to M \to P_0 \to P_1 \to \cdots \]
This leads to a cochain complex
\[ \text{Hom}_R(P_0, N) \to \text{Hom}_R(P_1, N) \to \text{Hom}_R(P_2, N) \to \cdots \]

Its cohomology depends only on \( M \) and \( N \), not on the choice of \( P_i \).

\[ H^0 = \text{Ext}_R^n(M, N) \quad \Delta \cong 0 \]

If \( R, M \) and \( N \) are all graded, then so is \( H^0 \).

In our case:

- \( R = A \) (graded)
- \( M = H^* X \)
- \( N = \mathbb{Z}/2 \)
- \( P_0 = H^* (\Sigma^{\infty} L_0) \)
\[ \text{Hom}_R(P_0, N) = \text{Hom}_A \left( H^{1/2}L_0, 2/2 \right) = \pi_0(L_0) \]

\[ X = X_0 \xrightarrow{g_0} X_1 \xrightarrow{g_1} X_2 \xrightarrow{g_2} \cdots \]

Adams resolution

Technical Lemma. Suppose \( X \) is connective and of finite type (each \( \pi_0 X \) is finitely generated). Let \( \hat{X} \) be the cofiber of \( \varprojlim X_0 \rightarrow X \rightarrow \hat{X} \). Then \( \hat{X} \) is the \( 2 \)-adic completion of \( X \), i.e.
\[ \tilde{\pi}_* X = \tilde{\pi}_* X \otimes \mathbb{Z}_2. \]

where \( \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \) -adic integers.

Theorem (Adams 1959) There is a spectral sequence converging to \( \tilde{\pi}_* X \) with

\[ E_2^{s,t} = \text{Ext}_A^{s+t} (H^s(X), \mathbb{Z}/2) \]

\[ d_r : E_r^{s,t} \rightarrow E_r^{s+r, t+r-1} \]

\( E_\infty \) is a subquotient of \( \tilde{\pi}_* (X) \).
The Adams filtration on $\pi_\ast X$ is defined as follows: An element $x \in \pi_k(X)$ has filtration $\geq s$ if it is lifted by a map which can be factored through $s$. 

$$\xymatrix{ R_k \ar[dr]_{x} & \ar[r] & X \ar[dl]^{x} \ar[l]_{f} \ar@{>->}[dl] \ar@{>->}[d]_{f_0} \ar@{>->}[r]_{f_1} & X \ar[l]_{f_2} \ar[r] & Y_1 \ar[l]_{f_3} \ar[r] & \cdots \ar[r] & Y_{k-1} \ar[r] & Y_k \ar[l]_{f_{k-1}} \ar[r] & \cdots \ar[r] & Y_1 \ar[l]_{f_2} \ar[r] & X }$$

with $H^*(X, R) = 0$

Example 1: $X = S^0$ and $x = \text{identity map} \ S^0 \to S^0$. 

(decreasing)
Since $H^s(L) \neq 0$, the above does not occur for any $s > 0$.

$x = 2^s$

$S_0 \xrightarrow{2} S_0 \xrightarrow{2} S_0 \xrightarrow{2} \cdots \xrightarrow{2} S_0$

$s$ factors

This map has filtration $s$. 