Browder's Ann. There is a framed closed mfd with Renne's invariant $I$ in $dim 2^{i+1} - 2 \Rightarrow h_j^2$ is a permanent cycle in the Adams $S$ (It can happen only in three dims)

If $h_j^2$ is a permanent cycle, it reps a map $S^{2^{i+1} - 2} \rightarrow S^n$, $h_j \in Ext_{S^{2^{i+1}}} S^n$, $|h_j| = 2^{i+1} - 1$

$h_j$ is known to exist for $1 \leq j \leq 5$. 
The (HHR) of does not exist for $j > 7$.

**Strategy of proof**: Construct a ring spectrum $\Sigma$ (nonconnective) with 3 properties:

(i) if $j \leq 7$, then its image in $\pi_\ast \Sigma$ is nontrivial (DETECTION)

(ii) $\Sigma^{\wedge} \Sigma \cong \Sigma$ (PERIODICITY)

(iii) $\pi_{-2} \Sigma = 0$ (GAP)
(ii) + (iii) \Rightarrow \Pi_{254} \mathbb{S}^2 = 0. If \exists \theta, its
image in this gp is nontrivial by (i).

How to construct \mathbb{S}^2 ??? We
need 2 tools

1. MV-theory (complex cobordism)
   both to construct \mathbb{S}^2 and to
   prove (i).

2. Equivariant stable homotopy
theory needed to prove (ii) and (iii).
An introduction to MV-theory

Let $G_{m,k}$ denote the space of complex $n$-planes in $C^{n+k}$. It is a compact complex analytic manifold of complex dimension $nk$. It is also a complex projective variety. $H^*(G_{m,k};\mathbb{Z})$ is known.

One has maps $G_{m,k} \to G_{m,k+1}$ induced by $C^{n+k} \to C^{n+k+1}$. 
Let \( \Omega^U(n) = \lim_{\longrightarrow \kappa} G_{n, k}^\mathbb{C} = G_{m, \infty}^\mathbb{C} \)
\( \Omega^B(n) \) in the real case

= space of complex \( n \)-planes in \( \mathbb{C}^n \).

There is a \( C^\infty \)-bundle over \( G_{m, k}^\mathbb{C} \)
with total space 
\[ E_{n, m, k} = \{ (z, x) \in \mathbb{C}^{n + k} \times G_{m, k}^\mathbb{C} \mid z \in \mathbb{C}^n \sim G_{m, k}^\mathbb{C} \} \]

There is a map 
\[ E_{n, m, k} \rightarrow G_{m, k}^\mathbb{C} \]

where 
\[ \rho^{-1}(x) \cong \mathbb{C}^n \]. Let \( M_{m, k}^1 \) be its one
point compactification, (Thom space)

\[ M\text{U}(n) = \lim_{\to k} M\text{U}_k \]

Properties:

a) \( H^*(\text{BU}(n); \mathbb{Z}) = \mathbb{Z} \langle c_1, c_2, \ldots, c_n \rangle \)

where \( c_i \in H^{2i} \) are Chern classes

which can be defined geometrically.

b) \( H^{k+2n}(\text{MU}(n); \mathbb{Z}) \cong H^k(\text{BU}(n); \mathbb{Z}) \)

Under the map \( \text{BU}(n) \to \text{MU}(n) \),
The image in $H^* \mathbb{C}_n$ is $(C_n)$. 

a) $H^*(BO(n); \mathbb{Z}/2) = \mathbb{Z}/2 [w_1, \ldots, w_n]$ 

where $w_i \in H^i$ is the $i$th Stiefel-Whitney class, which can be defined geometrically.

b) Under the map $BO(n) \rightarrow MD(n)$ 

the image in $H^*(\mathbb{Z}/2)$ is $(\mathbb{Z}/2)$. 

$H^{k+n}(MD(n); \mathbb{Z}/2) \cong H^k(BO(n); \mathbb{Z}/2)$
The \( \text{MU} \) prespectrum is defined by
\[
\text{MU}_{2n} = \text{MU}(n)
\]
\[
\text{MU}_{2n+1} = \Sigma \text{MU}(n)
\]
We need a map \( \Sigma^2 \text{MU}(n) \to \text{MU}(n+1) \)

We have maps \( g_{k,n} : G_{n+k} \to G_{n+1+k} \) induced by \( C_n \to C_{n+k+1} \)

This leads to a map \( BU(n) \to BU(n+1) \)

\( BU(n) \) has a \( C^n \)-bundle over it.
Induced vector bundles

Let \( p : E \to Y \) be a \( C^n \)-bundle over \( Y \). This means \( f'(y) = C^n \) for each \( y \in Y \) with certain conditions, and let \( f : X \to Y \) be any map. Let

\[
E' = \{ (x, e) \in X \times E : f(x) = p(e) \in Y \}
\]

\( E' \) is a \( C^n \)-bundle over \( X \), the bundle induced by \( f \).

We also get a map between the
one point compactifications (Thom spaces) of $E^*$ and $E$.

**Example** \[ BU(n) \xrightarrow{f} BU(n+1) \]

$BU(n+1)$ has a $C^{n+1}$-bundle $\gamma^{n+1}$

$BU(n)$ has a $C^n$-bundle $\gamma^n$

and a $C^{n+1}$-bundle $f^*\gamma^{n+1}$

We find $f^*(\gamma^{n+1}) = \gamma^n \oplus \varepsilon$

where $\varepsilon$ is the trivial $C^1$-bundle.
The Thom space is $\Sigma^2 \text{MU}(n)$

$\cong \Sigma^2 (\text{Thom space for } \mathbb{R}^n)$

induces the desired map

$\Sigma^2 \text{MU}(n) \to \text{MU}(n+1)$

This is the $(2n+1)$th structure map for the spectrum $\text{MU}$.

$\text{MO}$ is the prespectrum defined by

$\text{MO}_n = \text{MO}(n)$ with similar structure map.
Properties of MU and MO.
Both are ring spectra

\[ \pi_* \text{MU} = \mathbb{Z} \left\{ x_1, x_2, x_3, \ldots \right\} \quad x_i \in \pi_{2i} \]

\[ \pi_* \text{MO} = \mathbb{Z}/2 \left\{ y_2, y_4, y_6, y_8, \ldots \right\} \]

\[ y_i \in \pi_i \quad i \neq 2^{n-1} \]

\[ H_* \left( \text{MU} \wedge \mathbb{Z} \right) = \mathbb{Z} \left\{ b_1, b_2, \ldots \right\} \quad b_i \in H_{2i} \]

\[ H_* \left( \text{MO} \wedge \mathbb{Z}/2 \right) = \mathbb{Z}/2 \left\{ \overline{b}_1, \overline{b}_2, \ldots \right\} \quad \overline{b}_i \in H_i \]
Recall if $E$ is a prespectrum

\[
\begin{align*}
H_i(E) &= \lim_{\to n} H_{i+n}(E_n) \\
\Pi_i(E) &= \lim_{\to n} \Pi_{i+n}(E_n)
\end{align*}
\]

Then, in a space or spectrum $X$, we define $MU_*(X) = \Pi_m(X \wedge MU)$, the $MU$-homology or complex bordism of $X$, and $MU^nX = [X, \Sigma^nMU]$. 
the $MU$-cohomology on complex cobordism of $X$. Similar def. for $MU$.

These have similar formal properties to $H_\ast$ and $H^\ast$. In particular for a space $X$, $MU_\ast X$ has cup products.

$\mu_\ast (S^0) = \Pi_\ast (MU \wedge S^0) = \Pi_\ast (MU)$

$= as~before$

$MU^n (S^0) = \Sigma S^0, \Sigma^n \mu U] = \Sigma S^{-n}, \mu U]$
stack spectrum $= \pi_*^{m}(MU)$.

$MU^*(S^0) = 2 \left\{ x_1, x_2, \ldots \right\}$, $x_i \in MU^{-2i}(S^0)$.

For a space $X$, $MU^*(X)$ is an algebra over $MU^*(pt)$.

Let $X = \mathbb{C}P^\infty = \text{infinite dimensional complex projective space}$

$= BU(1)$.

$H^*(\mathbb{C}P^\infty; \mathbb{Z}) = \mathbb{Z}[x]$, where $x \in H^2(-)$

$= H^*(pt)[x]$.
$MU^* (\mathbb{C}P^\infty ; Z) = MU^*(k[x]) [[x]]$ where $x \in MU^*(\cdot)$

$MU^* (\mathbb{C}P^n) = MU^*(k[x])[x]/(x^{n+1})$

$\mathbb{C}P^\infty = \varprojlim \mathbb{C}P^n$

$MU^* (\mathbb{C}P^\infty) = \varprojlim MU^* (\mathbb{C}P^n)$

$= MU^*(k[x]) [[x]]$

a.g.

$X + x_1 x^2 + x_2 x^3 + \ldots \in MU^2 (\mathbb{C}P^\infty)$

$|x| = 2$ \quad $|x_1| = -2i$ \quad $|x_1 x_1| = 2$