Kervaire invariant problem

If $M^{4k+2}$ is a smooth framed manifold, can $\Phi(M)$ be nonzero?

Early answers:
- Yes for $k = 0, 1, 3$ ($y = 1, 2, 3$)
- No for $k = 2$ (Kervaire 1960)

This led to a nonsmoothable 10-manifold.

1967 Brown-Peterson
NO for $k > 0$ and $k$ even

1969 Browder
YES only when $k = 2^v - 1$
(assuming $4k + 2 = 2^{v+1} - 2$)

1970–1984 Barnatt, Jones, Mahowald, and Tangora
YES for $k$ as above with $j = 4, 5$

2009 HHR
NO for $j > 7$ ($j = 6$ still open)

Browder gave a criterion for $k = 2^v - 1$, it involves the
Adams spectral sequence and Steenrod operations.

Some facts from homotopy theory:

Let $A$ be an abelian (e.g., $\mathbb{Z}/2$) and $n$ a positive integer. Then there is a space $K(A, n)$ (Eilenberg-Mac Lane space) such that

$$H^n(X; A) = [X, K(A, n)]$$

is homotopy classes of maps.
This set has a natural abelian group structure due to nil properties of \( K(4, n) \).

Examples:

\[
K(\mathbb{Z}, 1) = \mathbb{Z}
\]

\[
K(\mathbb{Z}, 2) = \mathbb{C}P^\infty
\]

\[
K(\mathbb{Z}/2, 1) = 1R P^\infty
\]

Let \( K_n = K(\mathbb{Z}/2, n) \) and \( H^*(X) = H^*(X, \mathbb{Z}/2) \)

Suppose \( \sigma \in H^{n+k}(K_n) \)
This gap is known in all cases

\[ H^{n+k}(K_n) = \{ K_n, K_{n+k} \} \]

Let \( x \in H^n(X) = \{ X, K_n \} \)

\[ X \xrightarrow{x} K_n \xrightarrow{\sigma} K_{n+k} \]

The composition \( \sigma \circ x \) is an element in \( H^{n+k}(X) \) i.e. \( \sigma \) induces a map

\[ H^n(X) \rightarrow H^{n+k}(X) \]

This is a cohomology operation
It is natural in $X$, i.e. given

$$W \xrightarrow{f} X$$

$$H^nW \xleftarrow{f^*} H^nX$$

diagram commutes

$$H^{-n+k}W \xrightarrow{g^*} H^{-n+k}X$$

0 may or may not be a homomorphism

Facts about $K_n$

$$K_n \cong \Sigma_2 K_{n+1} = \text{Map}_x(\Sigma_1, K_{n+1})$$
$\sum K_n \xrightarrow{\phi_n} K_{n+1}$, related to $c_n$

$H^n(K_n) = [K_n, K_n] \cap \text{identity} = c_n$

$\exists \phi_n$ generated by $c_n$

$[\sum K_n, K_{n+1}] = H^{n+1}(\sum K_n)$

$= H^n(K_m) = \mathbb{Z}/2$

$\phi_n$ induces a map

$H^{n+k+1}(K_n) \xrightarrow{\phi_n^*} H^{n+k+1}(\sum K_n) = H^{n+k} K_n \cap \sigma$
Thm: $\sigma: H^n(X) \to H^{n+k}(X)$ is a hom. in $K_n$ if $\sigma$ is in image of $K_n$.

If $\sigma$ is in this image, it is also in the image of a similar map $H^{n+k+\ell} K_{n+\ell} \to H^{n+k} K_n$ for any $\ell > 0$.

Such a $\sigma$ is a stable cohomology operation.

Example where $k = n$. 
\[ H^n X \xrightarrow{\Delta^m} H^{2n} X \]

\[ X \xrightarrow{\cdot (x+y)^2} x^2 + y^2 \]

Since it is additive it is stable, i.e., related to similar operations.

\[ H^{n+x} (X) \xrightarrow{\Delta} H^{2n+x} (X) \text{ for all } t \geq 0. \]

Steenrod squaring operation.

e.g., \[ H^* K_i = \frac{2}{2} [x] \quad x \in H^i \]
Properties of the $A_n$:

i) $A_n^0 = 1 = \text{identity}$

ii) Cartan formula

$$A_n^m(xy) = \sum_{0 \leq i \leq m} A_n^i(x) A_n^{m-i}(y)$$

iii) Adem relation. Let $0 < a < 2b$

$$A_n^a A_n^b = \sum_{j=0}^{\lfloor b/2 \rfloor} (b-1-j) A_n^{a+b-2j} A_n^j$$

E.g. $A_n^1 A_n^{2n} = A_n^{2n+1}$
$A_q' A_q^{2n+1} = 0$

$A_q^n$ is decomposable unless $n = 2^k$ for some $k$

**Def** A monomial $A_q^{i_1} A_q^{i_2} \ldots A_q^{i_m}$ is admissible if

\[ i_1 = 2i_2, \quad i_2 > 2i_3, \quad \ldots \quad i_{m-1} > 2i_m \]

Using the Adem relation any monomial is a sum of admissible ones.
Then Cmy stable operation can be written in terms of the \( \hat{A}_q \)'s.

We get an algebra \( A_2 \) of stable mod 2 cohomology operation.

This is the mod 2 Steenrod algebra.

The admissible monomials form a basis for \( A_2 \).
$H^*(X)$ is a module over $A_2$, subject to the unstable condition
\[ A^m_n x = \begin{cases} x^2 & \text{if } x \in H^n \\ 0 & \text{if } |x| < n \end{cases} \]

\[ A_2 A_1 = A_1 A_2 \quad A_1^2 A_2 = 0 \]
\[ A_3 A_3 = A_3 \]
\[ A_2 A_1 \text{ is admissible} \]
Milnor's formulation

Consider \( A_n = \text{Hom}(A_2, \mathbb{Z})_{2/2} \)

The Cantor formula leads to a coproduct

\[
A_2 \rightarrow A_2 \otimes A_2
\]

\[
A_2^n \rightarrow \sum_{0 \leq i \leq n} A_2^i \otimes A_2^{n-i}
\]

This extends to an algebra map
Structure of $A_+$

$\Xi/2 \subseteq \{ \xi_1, \xi_2, \xi_3, \ldots \}$

$|\xi_i| = 2^i - 1$

In $A$, we have

multiplication

not commutative $\; A \times A \rightarrow A$

co-multiplication $\; a \rightarrow A \rightarrow A$

Finally, we have
not a commutative \( A_x \otimes A_x \rightarrow A_x \)

commutative \( A_x \otimes A_x \rightarrow A_x \)

\[
\sum_{0 \leq i \leq n-1} \mathbb{Z}_{n-1} \otimes \mathbb{Z}_i
\]

where \( \mathbb{Z}_0 = 1 \)

This is equivalent to Alden's formula.
A question: Is there a map

\[ S^{n+2i-1} \to S^n \quad \text{for } n \geq 0 \]

such that \( A \mathbb{Z}^{2i} \) acts nontrivially in \( H^{n+2i} \mathcal{X} \)?

The cofibre is \( S^n \cup C^{n+2i} = \mathcal{X} \).

\[ H^n \mathcal{X} \xrightarrow{\mathbb{Z}^{2i}} H^{n+2i} \mathcal{X} \]

\[ \mathbb{Z}/2 \to \mathbb{Z}/2 \]
Examples where the action is continuous:

\[ i = 0 \quad S^1 \to S^1 \]

\[ i = 1 \quad S^3 \xrightarrow{\nu} S^2 \quad \text{Hopf map} \]

\[ i = 2 \quad S^7 \xrightarrow{\nu} S^4 \]

\[ i = 5 \quad S^{15} \to S^8 \]

Thm (Adams 1961) You cannot do this for \( i > 3 \).