Recall Quillen Thom 1969. The topological FGL over \( \mathbb{R}^x \) MV is isomorphic to Jazards universal example.

Let \( F \) be a FGL over a torsion free ring \( R \).

Then over \( R\mathbb{Q} \), \( F \) is isomorphic to the additive FGL, i.e., there is a power series \( y(x) \in R\mathbb{Q}[\mathbb{Q}[x]] \) such that
\[ g(F(x, y)) = g(x) + g(y). \] \( g(x) \) is the logarithm of \( F \).

Example. Let \( F(x, y) = x + y + xy \)

\[ 1 + F(x, y) = (1 + x)(1 + y) \]

\[ \ln (1 + F(x, y)) = \ln (1 + x) + \ln (1 + y) \]

and

\[ \ln (1 + x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \]

\( g(F(x, y)) = g(x) + g(y) \).

Then (Maschenko) \( \ln \) the \( F \) \( C \) \( L \) over \( T_2n-2(\mathbb{C}) \)

\[ \log x = \sum_{n=1}^{\infty} \frac{\text{CP}^{n-1}x^n}{n} \] where \( \text{CP}^{n-1} \in T_2n-2(\mathbb{C}) \).
Given a FGL \( F \), define power series \( [\ell_n]_F (x) \) for integers \( n \) as follows:

\[
\begin{align*}
[1]_F (x) &= x, \\
[0]_F (x) &= 0, \\
[\ell_{n+1}]_F (x) &= F (x, [\ell_n]_F (x)), \\
[\ell_{m+n}]_F (x) &= \pi ([\ell_m]_F (x), [\ell_n]_F (x)).
\end{align*}
\]

**Example:**

1. \( \ell_1 (x) = (1 + x)^{-1} - 1 \)

\[
= \sum_{k=0}^{\infty} \left( \begin{array}{c} -1 \\ k \end{array} \right) x^k
\]
Lemma. For a prime $p$, over $\mathbb{F}_p$

\[
    [p]^n(x) = \begin{cases} 
    a_1 x^n + \text{higher terms mod } p & \text{for } a \neq 0 \\
    0 & \text{mod } p 
    \end{cases}
\]

\[\text{in } \mathbb{F}_p \quad [p](x) = x^p \quad \text{mod } p\]

If $F(x, y) = x + y$, then $[m]^n(x) = nx$

so $[p]^n(x) = 0 \quad \text{mod } p$.

Def. The integer $n$ is the height of $F$ at $p$.

When $[p]^n(x) = 0$, the height is $\infty$. 

Then over an algebraically closed field of char. $p$, two FGLLs are isomorphic if they have the same height.

Example:

$$\log x = \sum_{i=0}^{\infty} \frac{x^{p^i}}{p^i} = g(x)$$

Honda

$$= x + \frac{x^{p^2}}{p^2} + \frac{x^{p^3}}{p^3} + \ldots$$

This is the log of a FGLL over $\mathbb{Z}_p$ with height $n$ at $p$.

$$g^{-1}(g(x) + g(y)) \in \mathbb{Z}(p)[[x,y]]$$
Remark: If $E$ is an elliptic, we can choose a local co-ord $x$ at the identity element $x_0$. The group law for $E$ is a FGL near $O$. Its height at a reasonable prime in $O$ is $1$ or $2$.

Recall the power series $[n]_F(x)$ for $n \in \mathbb{Z}$. It can be regarded as an endomorphism of $F$, i.e.

$$[n] (F(x,y)) = F ([n](x), [n](y))$$
If \( F \) is defined over a \( \mathbb{Z}_p \)-algebra or a \( \mathbb{Z}_p^\times \)-algebra, we can find power series \( [n]_F(x) \) for \( n \in \mathbb{Z}_p \) or \( n \in \mathbb{Z}_p^\times \) with similar properties.

Thus we get homomorphisms

\[
\mathbb{Z}_p^\times \to \text{End}_F(F) = \text{endomorphism ring of } F
\]

with

\[
[n]_F(x) = nx \mod (x^2)
\]
Suppose $F$ be defined over an $A$-algebra, where $A$ is the ring of integers in a number field or a finite extension of $Q_p$. Can we make sense of $[a]_{f, e}(x)$ for $a \in A$?

Def: If we can, we say $F$ is a formal $A$-module.

Lemma (Ihara-Tate 1965) Let $A$ be the ring of integers in a finite extension of $Q_p$ with maximal ideal $\mathfrak{p}$ and $A/\mathfrak{p} = \overline{F}$.

Let $f(x) \in A[x]$ with
i) \( f(x) = \pi x \mod (x^2) \)

ii) \( f(x) = u x^6 \mod (\pi) \) for a unit \( u \)

\( e.g. \ f(x) = \pi x + x^3 \)

Then there is a formal \( A \)-module \( F \) over \( A \) for which \( \tilde{m}(x) > f(x) \).

They use this to construct field extension of \( K \), namely

\[
K_n = K [\pi^n] / (\tilde{m^n}(x) / x)
\]

\[
\text{Gal} [K_n : K] = [A / (\pi^n)]^\times
\]
Example

\[ K = \mathbb{Q}_2 \left[ \sqrt{37} \right] / (5^{4,1}) \]

\[ = \mathbb{Q}_2 \left[ \text{eight roots of unity} \right] \]

\[ A = \mathbb{Z}_2 \left[ \sqrt{37} \right] / (5^{4,1}) \]

\[ \Pi = 5^{-1} \]

and \[ \Pi^4 = 2, \text{ unit} \]

Let \[ g(x) = \sum_{i=0}^{\infty} \frac{x^{2^i}}{\Pi^{2^i}} = x + \frac{x^2}{\Pi} + \frac{x^4}{\Pi^2} + \frac{x^8}{\Pi^3} + \ldots \]

This is the log of a formal \( A \)-module \( \Pi \)

over \( A \).

\[ [\Pi] (x) = x^2 \mod \Pi \]

height is 4.

\[ [\Pi \Pi] (x) = x^8 \mod \Pi \]
Recall the Honda height in $FG_\mathcal{L}$
\[
\log = \sum_{i=0}^{p^n-1} \frac{1}{i}
\]
Consider this over $\mathbb{F}_{p^n}$. Its automorphism $g_P$ is $S_n$, the $n$th Morava stabilizer $g_P$.
There is a ring $\mathbb{W}(\mathbb{F}_{p^n})$, the Witt ring for $\mathbb{F}_{p^n}$.
It is a degree $n$ extension of $\mathbb{Z}_p$ obtained by adjoining $(p^n-1)$ th roots of unity. It is maximal ideal is $(p)$ and its residue field is $\mathbb{F}_{p^n}$.
$\text{Ker} (\mathbb{F}_{p^n}: \mathbb{F}_p) = C_n$
generated by the Frobenius map $x \mapsto x^p$. This automorphism lifts to $W$ and is denoted by $x \mapsto x^q$ where $x^q \equiv x^p \mod p$.

Let $E = W \langle \sigma \rangle / \langle \sigma^q - 1 \rangle$

$s w = w^q s$ for $w \in W$.

Then $E \otimes \mathbb{Q}_p$ is a division algebra of rank $n^2$ over $\mathbb{Q}_p$.

Consider the group of units in $E$ \(U_n\) is isomorphic to the automorphism
$\mathfrak{P}$ of $E \otimes E_{\mathfrak{P}}$.

Interesting property of $D$:

Any degree $n$ extension of $\mathbb{Q}_p$ is a subfield of $D$, e.g., $K$ as above embeds for $n = 4$, and the group has an element of order 8.