Toward the Detection Theorem

We need the Adams-Lurie case of the theorem for a “nice” ring spectrum \( R \)

Recall, for a “nice” ring spectrum \( R \)

the pair

\[
(A, \mathcal{F}) = (\Pi_* R, \Pi_* R \otimes R)
\]

is a Hopf algebroid over \( K = \Pi_0 R \), i.e., it is a corgroupoid in the category of graded comm \( K \)-algebras. This means for any such algebras \( S \), the sets...
\[ \text{Hom}_{k\text{-alg}}(A, S) \quad \text{and} \quad \text{Hom}_{k\text{-alg}}(T, S) \]

are the object + morphism sets of a groupoid. There are 5 structure maps:

\[ n_1 \quad \alpha \quad n_1 \quad \eta \quad \iota \]

\[ \Delta \quad \Theta \quad \Gamma \]

dual to source

identity

composition

composable morphism pairs
The $\mathbb{F}_2$-term of the $\mathbb{R}$-based Adams SS for $X$ is

$$\text{Ext}_R \left(A, \mathbb{R}_X(x) \right) = : \text{Ext} \left(\mathbb{R}_X(x) \right)$$

Two examples

1) Classical case $R = H/\mathcal{F}$

$$\pi_+ R = \pi_0 R = K = \mathbb{Z}/2$$

$$\pi_+ R^\wedge R = A_+ = \text{dual Steenrod alg}$$

$\eta_R = \eta_L$ and we get a Hopf algebra over $\mathbb{Z}/2$

$$|\eta| = 1 \quad |b_2| = 2$$

2) $R = MU$, $\pi_0 R = \mathbb{Z}$
\[ A = T \times MU = \mathbb{Z}[x_1, x_2, \ldots] \quad J = \text{Lagrangian ring of} \quad L \]

\[ \text{Hom}_{\text{z-alg}}(A, S) = \text{set of FGts } / S \]

\[ \Gamma = \pi_1^* MV \times MV = MU \times (MU) = \mathbb{L}[b_1, b_2, \ldots] \quad J = \mathbb{L}B \]

\[ \text{Hom}_{\text{z-alg}}(\mathbb{L}B, S) = 3 \text{ pairs of FGts } F, F' \text{ over } S \text{ with a } \]

\[ f(x) = x + b_1 x^2 + b_2 x^3 + \cdots \]

\[ f'(F(x, y)) = F'(f(x), f(y)) \]

The groupoid is that of FGts and \textit{struct isomorphisms} between them.
To describe the structure maps, we tensor with $\mathbb{Q}$

$$\log x = \sum m_i x^{i+1}, \quad \text{where} \quad m_i = \frac{[\Delta^i]}{i+1}$$

$$e(\lambda \otimes \Delta[[x]])$$

$$c(b_i)$$ is defined recursively by

$$\sum_{i \geq 0} c(b_i) \left( \sum_{j \geq 0} b_j (x)^{i+j} \right) = 1$$

where $b_0 = 1$

follows

$$f^{-1}(f(x)) = x$$

$$f^{-1}(x) = \sum c(b_i) x^{i+1}$$

$$f(x) = \sum b_i x^{i+1}$$
Remark 1) For the Steenrod algebra consider \[ f(x) = \sum_{i \in \mathbb{Z}} x^{2^i} \]

\[ f^{-1}(x) = \sum_{i \in \mathbb{Z}} c(2^i) x^{2^i} \]

Can also derived the Milnor isoproduct this way.

Remark 2) \[ c(b_i) \in \mathbb{Z} [b_1, b_2, \ldots ] \]

We do not need more of \( \mathbb{P} \times \mathbb{MV} \) here. This related to the fact for each \( \mathbb{Z} \)-algebra \( S \) the groupoid is split. The group is \( \text{Hom}(B, S) \).
$B = \exists \Sigma b_1, b_2 \ldots, l$

equipped the configuration $c$ defined above.

Back to our structure:

$\eta_L : L \rightarrow LB$ standard inclusion
$\epsilon : LB \rightarrow L$ $b_i \mapsto 0$ for $i > 0$

$\eta_R$ defined over $A$ by

$$\sum_{i \geq 0} \eta_R(m_i) = \sum_{i \geq 0} m_i \left( \sum_{i \geq 0} c(b_i) \right)^{i+1}$$

$\Delta$ defined by

$$\sum_{i \geq 0} \Delta(b_i) = \sum_{i \geq 0} \left( \sum_{i \geq 0} b_i \right)^{i+1} \otimes b_i$$
The relation between the $m_i$ and the $\chi_i$ is complicated.

For computational purposes, localize at a prime $P$, and replace $MV$ by $BP$, the Brown-Peterson spectrum.

Def A FG L over a torsion free ring $R$ is $\phi$-typical if its logarithm has the form $\log \chi = \sum_{i \geq 0} l_i x^{p^i}$, $l_0 = 1$. 
Thm (Cartier 1960s) Any FG Flat over a \( \mathbb{F}_p \)-algebra, is canonically iso to a - typical one \( G \).

If \( \log_{G} x = \sum_{i \geq 0} m_i \cdot x^{p^i} \)

then \( \log_{G} x = \sum_{i \geq 0} m_i \cdot x^{p^i} \).

Thm (Brown-Peterson 1967)

\[ \text{MU}_{(p)} = V \cdot \Sigma^2 \text{BP} \]

\[ \Gamma_{(p)} \text{BP} = \mathcal{E}_{(p)} \left[ U_1, U_2, \ldots \right] \text{ s.t. } |U_i| = 2(p^i - 1) \]
Thm (Quillen 1969) BP is a Nilring spectrum with the following structure:

\[ BP_* \otimes \mathbb{Q} = \mathbb{Q}[l_1, l_2, l_3, \ldots] \]

\[ |l_i| = |v_i| = |t_i| = 2(\phi_i - 1) \]

\[ \Gamma = BP_* \left( BP \right) = BP_* \left[ x_1, x_2, \ldots \right] \]

\[ \eta^R (l_n) = \sum_{0 \leq i \leq n} l_i \cdot t_i^{n-i} \quad l_0 = t_0 = 1 \]

\[ \Delta \text{ is determined by} \]

\[ \sum_{i \geq 0} l_i \cdot \Delta (t_i) \cdot \phi_i = \sum_{i \geq 0} l_i \cdot t_i^{n+i} \otimes \phi_i \]

(more explicit than \( \eta^R (m_n) \))
Remark: This Hopf algebroid is not split, unlike $MU_x(MU)$.

We have the Adams-Novikov $SS$ based on $BP$-theory.

The relation between the $\nu_i$ and $l_i$ is not complicated. One choice of the $\nu_i$ due to Hatcher is given recursively by

$$
\mu l_n = \sum_{0 \leq i < n} l_i l_i^* \nu_{n-i}^{-1}, \quad l_0 = 1
$$
This leads to

\[ l_1 = v_1 / p \]

\[ l_2 = v_2 / p + v_1^{1+p} / p^2 \]

\[ l_3 = v_3 / p + (v_1 v_2 + v_2 v_3) / p^2 + v_1^{1+p+p^2} / p^3 \]

etc.

We need to know \( \eta_R (v_i) \)

We have nice formula for \( \eta_R (l_i) \)

\[ \eta_R (v_1) = v_1 + pt \]

\[ \eta_R (v_2) = v_2 + v_1 t_1 p - v_1 p t_1 \quad \text{mod } p \]
\[ \triangle(x_1) = x_1 \otimes 1 + 1 \otimes x_1 \]

\[ \triangle(x_2) = x_2 \otimes 1 + 1 \otimes x_2 + 1 \otimes 1 \mod (\nu_i) \]

We will need to know certain elements in \( \operatorname{Ext}^1(B\mathbb{P}_x) \) and \( \operatorname{Ext}^2(B\mathbb{P}_x) \).

\( \theta \) is related to \( \operatorname{Ext}^{2,2i+1} B\mathbb{P}_x \).

Before we said \( \theta \) was related to \( h_{i \ast} \in \operatorname{Ext}^{2,2i+1} A \) \( (2/2, 2/2) \).
There is a relation between the Novikov and Adams Ext groups.

One has maps of ring spectra

\[ \text{MU}\mathbb{Z}_2 \longrightarrow \text{H}/2 \]

\[ p = 2 \]

\[ \text{BP} \]

\[ p \text{ induces a map of Ext groups} \]

\[ h^2 \text{ is in the image of } p \] only

\[ h_0 \]
for $1 \leq j \leq 3$.

(This leads to a proof that $b_j$ for $j \geq 3$ is not a permanent cycle.)

Extra meetings:
- Thursday, 4/22, 3:30
- Friday, 4/30, 4:00