Proof of Alice Theorem is in 2 steps

1. Some formal arguments that reduce us to a special case called the Reduction Theorem.

For the trivial gap it always if we kill all tiny ghost MxMV in positive dimensions, we get H2.

There is an easy version which is not obvious.
2) Prove the Reduction Thm. This requires some calculation.

Some formalities

Properties of the norm function $N_H$

If $H \subset G$ and $X$ is an $H$-equivariant spectrum, then $X^{g \cdot h} = \frac{1}{|G|} \cdot X$, $g = |G|$, $h = |H|$

has a $G$-action by permuting the factors on the space level

$Map_H (G, x)$ is a $G$-space underlain in $X^{g \cdot h}$. 
Need a formula for $N^G_H(X_1 \vee X_2 \vee \ldots)$

Example: $G_1 = C_4 \quad H = C_2$

$X = S^0 \vee S^{p_2}$

$X^{(2)} = (S^0 \vee S^{p_2}) \setminus (S^0 \vee S^{p_2})$

$= (S^0 \setminus S^0) \vee (S^0 \setminus S^{p_2}) \vee (S^{p_2} \setminus S^0) \setminus (S^{p_2} \setminus S^{p_2})$

$N^4_2 X = S^0 \vee (C_4 \setminus C_2) \setminus S^{p_4}$

[For a rep $V$ of $H$, $N^G_H S^V = S^{\text{Ind}_H^G V}$]

$= N^4_2 S^0 \vee \text{(same)} \setminus N^4_2 S^{p_2}$
Let $\mathbf{X} = \bigvee_{g} X_{g}$. Each $X_{g}$ is an $H$-orbit.

Want to describe $N_{H}^{G} X$.

Let $K = \text{hom}(G/H, J)$ be a $G$-set.

For $k \in K$, let $H \subset C_{G}(k) \subset G$ be its stabilizer.

Let $[k]$ be the $G$-orbit of $k$. It is seen as a $G$-set to $G/H_{G_{k}}$.

$$X_{G} = \bigvee_{g} X_{g} \bigg|_{G_{G_{k}}}$$
\[ X_k = N \frac{G_k}{H} X_k \]

**Proposition**

\[ N \frac{G_i}{H} X = \bigvee_{k \in K/G} \frac{G_k}{G_i} X_k \]

\[ X = \bigvee_{j \in J} X_j \]

Some eigen ring spectra underlain by wedges of spheres

Let \[ H < G \]

\[ Y = \bigvee_{i=0}^{\infty} S^{i} \]

is an \( H \)-spectrum
with a multiplication

\[ \gamma \cdot \gamma = \sqrt{\frac{\sin \gamma \cdot \sin \gamma}{\sin \gamma \cdot \sin \gamma}} \rightarrow \gamma \]

This is associative but not comm. (??)

Let \( \bar{x} \in \prod_V S_0^V \) denote the inclusion

\[ S(V) \rightarrow S_0 [S_0^V] \quad S_0^V [\bar{x}] = S_0^V [S_0^V] \]

\( N^G \), \( S_0^V [\bar{x}] \) is a known quantity

Think of this as a "polynomial algebra" on one generator.
We can smooth such things for various $V$s and get polynomial algebras on several generators.

Recall $\Xi x \left[ MV_{\mathbb{R}} \right] = \mathbb{Z}[x_1, x_2, \ldots]$

For each $x_i$ there is an $\overline{x_i} \in \prod_{i=2}^\infty MV_{\mathbb{R}}$

which $i_0^* \overline{x_i} = x_i$

($i_0^* =$ forgetful functor or restriction to trivial gb)

This means that for each $i \geq 0$
there is a map $S^0 \left[ S^i \mathbb{R} \right] \to MUR$
representing all powers of $\mathbb{X}^i$

Let $S^0[\mathbb{X}^i] := S^0 \left[ S^i \mathbb{R} \right]$. 

Let $W \subset S^0[\mathbb{X},] \subseteq S^0[\mathbb{X}_2] \subseteq \ldots$
It is an associative ring spectrum
There is map $W \to MUR$
which is multiplicative refinement
of $\pi^U_*(MUR) = \pi^U_*(MU)$.

For $G = C_{2^k}$, then the map 
\[ A = \mathbb{Z}/2 \left[ W \mathbb{Z}/2 \right] \to \mathbb{Z}/2 \mathbb{MUR} = \mathbb{MU}(G/2) \]
in a mult. refinement of $\mathbb{M}U^{(1,2)}$.

More formal machinery:

How to construct "ideals" in the ring spectrum.

Fix a $q$-flat $G$, and let $\Sigma H^i = G_i^2$

with $V_i$ a rep of $H_i$. For each $i$

these we get $\Sigma [\overline{V}_i]$ and

$N_{H_i}^G \Sigma [\overline{V}_i] = \Sigma [\rho_i \overline{V}_i]$

We can smash these together and

get an "equiv. polynomial algebras"
How to construct ideals generated by sets of monomials

Let \( J = \prod_{i} G_{i}/H_{i} \), a \( G_{i} \)-set

\[ N_{0} = \mathbb{Z}_{\geq 0} \cup \{1, 2, 3, \ldots\} \]

\[ N_{0}^{-J} = \text{set of finitely supported functions } f: J \rightarrow N_{0} \]

also a \( G_{i} \)-set.

\[ V_{\phi} = f(1) v_{1} + f(2) v_{2} + f(3) v_{3} + \cdots \]

is a rep of the algebraic \( G_{i} \).
Then \( T = \bigvee_{f \in \mathcal{I}} S_{T^f}^{V_f} \)

\[
= \bigvee_{[b] \in N_0^J / G_0} \bigoplus C_{G_0} \mathcal{I} \bigoplus C_{G_0}^{V_f}
\]

\( N_0^J \) is a commutative monoid

An ideal \( \mathcal{I} \subseteq N_0^J \) is a subset

with \( \mathcal{I} + N_0^J = \mathcal{I} \)

For a \( G \)-invariant ideal \( \mathcal{I} \), let

\[
\mathcal{T}_\mathcal{I} = \bigvee_{f \in \mathcal{I}} S_{T^f}^{V_f}
\]
This an equiv sub-bimodule of $T$

$T \twoheadrightarrow T_J$

$T \rightarrow T_J$

Example

Let $\dim : \mathbb{N}_0^J \rightarrow \mathbb{N}_0$

$$
\dim f = \dim V_f = \sum_{j \in J} b(j) \dim V_j
$$

Define an ideal $I_d \subseteq \mathbb{N}_0^J$
\[ \text{Id} = \{ f \in \mathcal{W} : \dim f \geq d \} \]

Note: \( \mathcal{W} \) is \( C_1 \)-invariant

Let \( \mathcal{M}_d = T_{\text{Id}} = \bigcup_{\dim V \geq d} \]

\[ \mathcal{M}_d / \mathcal{M}_{d-1} = \bigcup_{\dim V \geq d} \]

Example of interest

\[ G = C_{2k}, \quad H_i = C_2 \quad \text{for} \quad i = 1, 2, \ldots \]

\[ V_i = i \mathbb{P}_2 \]

\[ M_i : S \rightarrow M \mathbb{V}_R \quad \text{a certain element} \]
\[ A = S^0 \left[ G \cdot M_1, G \cdot M_2, \ldots \right] \xrightarrow{J} N^2_2 \text{ MU} = \text{MU}(g/2) \]

is a null refinement of \( \Omega_2^* \text{MU}(g/2) \)

\( A \supset M_{2d} \) described above

\[ M_{2d} / M_{2d+2} = \text{wedge of slice cells} \]

Set \( K_{2d} = \text{MU}(g/2) \wedge M_{2d} \)

Note \( \text{MU}(g/2) \) and \( M_{2d} \) are both \( A \)-modules

\[ \text{MU}(g/2) = K_0 \supset K_2 \supset K_4 \supset K_6 \ldots \]
Jet $p^2_{\text{d}} \text{MU}(g/2) = \text{MU} g^2 / K_{2d+2}$

$\tilde{p}^0_{\text{MU}}(g/2) = K_0 / K_2$

**Reduction Thm**

$R(\infty) : = K_0 / K_2 = \frac{H}{2}$

**Formal fact**

$K_{2d} / K_{2d+2} = R(\infty) \cdot (\text{MU} g^2 / \text{MU} g_{2d+2})$

The layers in our new tower
\[ \mathfrak{P}^{2d} = \text{MU}(q/2) / K_{2d+2} \mathbb{Z} \] and as above. If we knew \( \text{R}(0) = 0 \), then we could identify \( \mathfrak{P}^{2d} \) with the slice tower.