Towards the proof of Reduction Thm.

Last time we reduced to Slice Thm to the RT.

For each $R > 0$ we have an associated ring spectrum $S^0 [S^{AR}]$ over $C_2$ and a map $\varphi_R : S^0 [S^{AR}] \to M \mathbb{V} \mathbb{R}$

Smashing these together for all $R > 0$ and applying the norm functor...
$N_{\ast}^g$ gives us a spectrum $A$ and an equivariant map $A \to MU(q_2)$

Recall (see notes from last time) we have a formula $N_{\ast}^g$ (wedge)

Our spectrum $A$ is a wedge of slice cells, and our map $A \to MU(q_2)$ is a multiplicative refinement of $\pi_{\ast}^u(MU(q_2))$. Consider the spectrum $G_7 = C_{2n}$ $g = |\alpha|$
\[ \text{Here } \text{MU}^{(q/2)} \text{ is an } A\text{-module via } \theta \]

\[ \exists \theta^0 \Rightarrow \text{via } \theta \]

the projection map \[ A \to \theta^0 \]

which is null on all positive dimensional summands of \[ A. \]

\[ R(\infty) \text{ is obtained (equivariantly)} \text{ from } \text{MU}^{(q/2)} \text{ by killing all of } \text{TMU}^{(q/2)} \]

in positive dimensions.
Reduction Thm: The map $R(\infty) \to HZ$ is a $G$-equivalence.

Remarks

1) This implies the Slice Thm as explained last time.

2) Statement is obvious if we forget the gp action.

3) It was proved for $G_1 = C_2$ by Hu-Kriz in 2001.

Why is $S^0 \wedge \mathbb{F}_{k^p}$ not commutative?
is commutative only up to homotopy

\[ \text{swap} \quad \Rightarrow \quad \text{skew} \]

\[ s_{k^2} \leq s_{p^2} \]

Reduction Theorem: The map \( R(\infty) \xrightarrow{\sim} HZ \)

is a \( G \)-equivalence.

Proof is by induction on \(|G|\).

For any proper subgroup \( H \subset G \), it is easy to check (see Section 4 of...
the HHR preprint online

\[ i^*_h R^*_c(\omega) = R^*_c(\omega) \] so we can assume \( i^*_h f_\omega \) is an \( H \)-equivalence.

We will smash \( f_\omega \) with the retraction separation sequence

\[ EC_2 \rightarrow S^0 \rightarrow EC_2 \]

\( G \) acts equivariantly

\( G \rightarrow G_2 \) trivially
\[ \mathbb{E}C_{24} \times R^n(\infty) \rightarrow R^n(\infty) \rightarrow \mathbb{E}C_{24} \times R^n(\infty) \]

\[ \downarrow f' \downarrow f \downarrow f'' \]

\[ \mathbb{E}C_{24} \times H2 \rightarrow H2 \rightarrow \mathbb{E}C_{24} \times H2 \]

\[ f' \text{ is an equivalence because } \mathbb{E}C_{24} \text{ is a free } \mathbb{Z}_2 \text{-complex. It suffices to show } f'' \text{ is a } G \text{-equivalence.} \]

Being a G-equiv. means that f'' induces an ordinary on the fixed set for each subgroup of G. We
already know this for every $\phi$, so we need only show it for $(E \mathbb{G} \times R(\alpha)) \rightarrow (E \mathbb{G} \times H(\mathbb{Z}))$

$\phi^* R(\alpha) \rightarrow \phi^* H(\mathbb{Z})$

**Facts about geometric fixed pts.**

$\pi^G_\alpha (E \mathbb{G} \times x) = \alpha^{-1} \pi^G_\alpha x$

For $x = H(\mathbb{Z})$, previous computations show this is
\[ 2/2 \sum_{a_{26}} \alpha \tau_{17} \]

\[ M_{26} \in \prod_{-2}^{G} \]

\[ G_6 \in \prod_{-6}^{G} \]

\[ b = a_6^{-2} m_{26} \in \prod_2 \]

Hence \[ \prod_x \hat{G} H_2 = 2/2 [b^7] \]

What about \[ \prod_x \hat{G} R(x) \]?

Recall \[ \hat{G} \text{MU}(q/2) = \text{MO} \]

\[ = \text{unoriented cobordism spectrum} \]

\[ \prod_x \text{MO} = 2/2 \sum h_2, h_4, h_5, h_6, \ldots \]

\[ h_k \in \prod_k \quad k \neq 2^\ell - 1 \]
An alternate description of $R(\infty)$

\[ \bigwedge^k \mu^{(g/2)} = \mathbb{Z} \quad \text{for} \quad k > 0, \quad 0 \leq j < g/2 \]

i.e. there are $g/2$ generators in each even dimension.

Let $R(m)$ be the spectrum obtained from $\mu^{(g/2)}$ by killing the first $m$ sets of generators. $R(\infty) = \lim_{m \to \infty} R(m)$

What is $\bar{H} R(m)$?

Support $m \neq 2^{t-1}$. Then there is
a cofiber sequence

\[ \Sigma^m \vartheta^G R(m-1) \xrightarrow{h_m} \vartheta^G R(m-1) \rightarrow \vartheta^G R(m) \]

Recall \( R(0) = MU(\mathbb{Q}) \) and \( \vartheta^G R(0) = MO \)

\( \vartheta^G R(m) \) is a relative of \( MO \).

For \( m = 2^k - 1 \) then we get

\[ \Sigma^m \vartheta^G R(m-1) \xrightarrow{0} \vartheta^G R(m-1) \rightarrow \vartheta^G R(m) \]

\( \vartheta^G R(m-1) \vee \Sigma^2 \vartheta^G R(m-1) \)
\[ m = 1: \Sigma M \xrightarrow{O} M \xrightarrow{\Phi} R(1) \]
\[ \Phi^2 R(0) \]
\[ M \cup \Sigma M \]

\[ m = 2: \Sigma \Phi^1 R(1) \xrightarrow{M_2} \Phi^1 R(1) \xrightarrow{\Phi} R(2) \]
\[ M \cup \Phi R(2) \]
\[ M_2 \cap \Sigma M \cup M_2 \]

\[ m = 3: \Sigma \Phi^2 R(2) \xrightarrow{0} \Phi^2 R(2) \xrightarrow{\Phi} R(3) \]
\[ M \cup \Phi^3 R(3) \]
\[ M_2 \cap \Sigma^2 M \cup M_2 \]

\[ \Phi^4 R(4) = M \cup (h_2, h_4) \cup (s^0 \cup s^2) \cup (s^0 \cup s^4) \]

...
\[ T_x \ (M_0) = \frac{2}{12} \left( h^2, h^4, h^6, h^8, \ldots \right) \]

We get \[ T_x \mathbb{E}^G R(0) = \frac{2}{12} \] in every even dimension.

**Conclusion:** \( T_x \mathbb{E}^G R(0) \) and \( T_x \mathbb{E}^G HZ \) are abstractly isomorphic. We still need to show the iso is induced by the map.

**Remark:** If this is going to work, \( R(n) \) cannot be a ring spectrum! If it were, then \( \mathbb{E}^G R(m) \) would be
one. But (we hope) the map
\[ \overline{G} R(m) \to \overline{G} R(\infty) \to \overline{G} H \]
has image in \( T_x \) which is not a subring.

How to define generators of \( T_x \text{MV}(q/2) \)
in a nice way.

Pick a \( C_2 \)-map \( \text{MV}_R \to \text{MV}(q/2) \)

\[ H_x \text{MV} = \mathbb{Z} \left[ m_k : k \geq 0 \right] \quad m_k \in H_{2k} \]

\[ \log x = x + \sum_{k \geq 0} m_k x^k \]
\[ \log x \in \pi_* \text{MU} \otimes \mathbb{Q} \quad \llbracket x \rrbracket \]

\[ e \in \pi_* (\text{MU}) \quad \llbracket x \rrbracket \]

\[ \pi_* \text{MU} \rightarrow \pi_* \text{MU} \rightarrow \pi_* \text{MU} \otimes \mathbb{Q} \]

\[ H_*^m \text{MU}(g/2) = \mathbb{Z} \left[ y^j m_r : r > 0, 0 \leq j < g/2 \right] \]

with \[ y^{g/2} m_r = (-1)^r m_r \]

\[ I = \text{ker} \pi_* \pi_*^m \text{MU}(g/2) \rightarrow \mathbb{Z} \]

\[ I_H = \text{ker} H_*^m \text{MU}(g/2) \rightarrow \mathbb{Z} \]
\[ Q_\ast = I / I^2 \text{ indecomposable in } \Pi_\ast \text{ MV}(q/2) \]
\[ Q_{H\ast} = I_{H\ast} / I_{H\ast}^2 \text{ indecomposable in } H_{H\ast} \text{ MV}(q/2) \]

Lemma (Milnor)

\[ Q_{2^k} \cong Q_{H2^k} \text{ if } k \neq 2t - 1 \]

For \( k = 2^t - 1 \) we have SES

\[ 0 \to Q_{2^k} \to Q_{H2^k} \to \mathbb{Z}/2 \to 0 \]

\[ y \text{ mod } 2 \to 1 \]
Lemma A. Let \( n = \sum a_j x^{j m} \in \mathbb{Q} H_{2k} \) with \( a_j \in \mathbb{Z}_2 \) for \( j \leq \sqrt{2} \) and \( x \neq 2^{1/2} \).

The map
\[
\mathbb{Z}_2 \Sigma G_7 \rightarrow \mathbb{Q} H_{2k} \cong \mathbb{Q}_{2k}
\]

factors thru a map
\[
\mathbb{Z}_2 \Sigma G_7 / (y^{1/2} - (-1)^k) \rightarrow \mathbb{Q} H_{2k}
\]

which is an isomorphism if
\[
\Sigma a_j = 1 \mod 2.
\]

Lemma B. For \( k = 2^{\ell - 1} \) and \( n \) as above, the map
\[ \mathbb{Z}/2 \mathbb{Z} \rightarrow \mathbb{Q} + \mathbb{Z}/2 \mathbb{Z} \]

then

\[ \mathbb{Z}/2 \mathbb{Z} \rightarrow \mathbb{Q} \]

is an iso if

\[ y = (1-x)^m \]

with \( m \) as before