**Prop 7.1**

The image of

\[ V = (2^k - 1) \mathcal{R}_0 \]

\[ \mathcal{R}_0 \] is not contained in that of \((1 - \mathcal{R}_0)\), but the image of \(y_i\) is not. [This implies the Reduction Theorem.]
Proof: \( \pi_0 \mathcal{E}_{2n+1}^m R(2^{l-2}) \) follows \( \sqrt{\mathcal{V}} \) from 7.13

\[
\begin{align*}
\pi_0 \mathcal{E}_{2n+1}^m R(2^{l-2}) & \subset \pi_0 \mathcal{E}_{2n+1}^m R(2^{l-2}) \\
\pi_0 \mathcal{E}_{2n+1}^m R(2^{l-2}) & \subset (1-\theta) \pi_0 \mathcal{E}_{2n+1}^m R(2^{l-2})
\end{align*}
\]

(\( \Box \))

**Lemma 7.13** Let \( M \) be a \( G \)-spectrum which is \( 0 \), i.e., built of slice cells of dimension \( 0 \).

Then
$\mathbb{C} \times \mathbb{M} \to \mathbb{C} \times \mathbb{M} \to \mathbb{G} \times \mathbb{M}$

The image of $\alpha$ is the same as that of transfer.

Proof: $\mathbb{G}$ is onto because in dim 0 the 2 spectra are the same. QED

This completes the proof of 7.12 and hence of the Reduction Theorem.

Recap
Browder's Thm 1969: A framed manifold with nontrivial Kervaire invariant must have dimension $2^{j+1} - 2$ for some $j \geq 1$. Such a manifold exists iff $h_j \neq 0 \in \text{Ext}^2_\mathbb{Z} \mathbb{Z}_{2j+1}$. 

This is a permanent cycle in the Adams SS.

An element in $\pi_{2^{j+1}} S^0$ representing $h_j$ (if it exists) is called $\Theta_j$. It was known to exist for $1 \leq j \leq 5$.

Thm (HHR 2009) $\Theta_j$ does not exist for $j \geq 7$.

The case $j = 6$ is still open.
Strategy of proof: Construct a map $S^0 \to S^2$ where $S^2$ is a spectrum with

a) Detection

If $\exists \theta_i$, its image in $T_{2^i} S^2$ is nontrivial.

b) Periodicity

$E$ is periodic $S^2 \cong S^2$.

c) Map

$\varpi_k S^2 = 0$ for $-4 < k < 0$. If $\exists \theta_i$, it has nontrivial in $\pi_{2^{i+4}} S^2$ by a)

$\pi_{2^{i+4}} S^2 = \pi_{2^i} S^2$ by b)

= 0 by c)
Our Alpine hike:

* For each $n \geq 0$ and each prime $p$ there is a spectrum $E_n$ (Morava $E$-theory) related to height $n$ formal group laws.

* There a profinite gp $S_n$ (with Morava stabilizers $G_p$), the automorphisms of a height $n$ $E_0$-algebra $\overline{T}$ act on $E_n$. The “action” is only defined up to homotopy, but it is possible to construct homotopy fixed point sets $\overline{E}_n^{hS_n}$ for any closed subgroup $G \leq S_n$. 
The case of finite $S$ of particular interest.

$S_n$ contains a cyclic subgroup of order $p^{r+1}$ iff $(p-1)p^{r} | n$. We were studying some such $E_n^p$. For $(p, n) = (2, 7)$ we have a cyclic subgroup of order 8.

We knew $E_7^2$ has detection and periodicity, but we could not prove a gap theorem. **SUMMER 2008**
An alternate approach

* $C_2$ acts on $MU$ by complex conjugation.

This makes $MU$ a $C_2$-equivariant spectrum.

This leads to a $C_{24}$-equivariant structure on $N^2 \pi MU = MU^{(2)}$, e.g. $MU^{(4)}$ is a $C_8$-spectrum.

A relative of $MU^{(4)}$ will be our substitute for $E_4$.

* An equivariant cyclic 2-quotient can be studied via its slice tower, which is an equivariant analog of the Postnikov tower.
Classically, $F^n X$, the $n$th Postnikov section of $X$, is the spectrum obtained by kill all homotopy groups above dimension $n$.

$$F^n X \longrightarrow X \longrightarrow F^{n+1} X$$

$n$-connected cover of $X$

$$F^n X \longrightarrow F^n X \longrightarrow F^{n+1} X$$

Eilenberg-Mac Lane spectrum which satisfies nontriviality gap $F^n X$.

The Postnikov tower is
\[ \cdots \xrightarrow{p^{n+1}} X \xrightarrow{p^n} X \xrightarrow{p^{n-1}} X \xrightarrow{p} X \xrightarrow{p} X \xrightarrow{p} \cdots \]

\[ \lim_{n \to \infty} p^n X = X \quad \text{and} \quad \lim_{n \to \infty} p^{-n} X = \ast \]

An equiv analog:

For each \( H \subseteq G \) and \( m \in \mathbb{Z} \), let

\[ \hat{S}(m, P_H) = G_{\cdot} \cdot \cdot \cdot \hat{S}^{m, P_H} \quad \text{for} \quad m = |H| \]

\[ \hat{S} = \bigvee_{m \in \{0, |G|/H\}} \hat{S}^{m, P_H} \]

This is a \( G \)-spectrum of "dimension \( m_H \)"
$p^n X$ is the $G$-spectrum obtained from $X$ by killing all $G$-equivariant maps to $X$ from $\tilde{S}(m^G)$ for $mh > n$ and from $\Sigma^r \tilde{S}(m^G)$ for $mh-1 > n$. This leads to a tower similar to the one above. Here $p^n X$ has rich $T^G$. The tower leads to a $SS$ with $E^s_2 = T^G_{t-s} P^0 X$ with $t \in RO(G)$.

$\Rightarrow \frac{\partial}{\partial s} X$
In the classical case $E_{2^{2m}} > 0$ only for $t = 2s$, and the $SS^2$ collapses from $E_2$. In every case it is interesting.

Alice Thom. Let $X = MU(2s)$. Then $P^0_{2s}X$ is

- contractible if $s$ is odd,
- $H \simeq \bigvee W$ where $W$ is a certain wedge of $S(m^p, n)$ for $m, h = s$ and $s$ is even.

This is a "formal" consequence of the Reduction Theorem.
What is this good for?
We can calculate $\prod_{\mathbb{R}^+} \mathbb{R}_{(\mathfrak{p}^+)}^{\mathbb{Z}}$

with relative ease. Each $(\mathfrak{s}((\mathfrak{p}^+))^{\mathbb{Z}})^G_{\mathfrak{p}^+}$

has the gap property (i.e. $\pi_{-2} = 0$)

for all $m \in \mathbb{Z}$ and all $h > 1$.

It follows for certain $x \in \prod_{\mathfrak{p}^{G_{\mathfrak{p}^+}}} \mathbb{M}_G(\mathbb{C})$

$(x^{-1} \mathbb{M}_h(\mathfrak{g}/\mathfrak{h}))^G$ also has the gap property

We can also show that

$(x^{-1} \mathbb{M}_h(\mathfrak{g}/\mathfrak{h}))^{\mathbb{C}}$ has periodicity
For appropriate \(G\) and \(x\),

\[(x^{-1} M^T v_{(g/2)})^h G\]

has detection property.

**Fixed Point Theorem**

\[(x^{-1} M^T v_{(g/2)})^h G = (x^{-1} M^T v_{(g/2)})^G\]

The simplest case with detection property is \(G = C_8\), \(x \in \Pi^G_{19} M^T v_{(g)}\).

Periodicity dimension is 256.
Historical note: The odd primary case.

There is no known odd primary analog to any manifold-theoretic statement. But there are odd primary analogs

\[ \psi^i \in \text{Ext}_A^2 \left( \mathbb{H}^{0 \times 2 \times 2 2^i-1} \right) \]

\[ \psi^i \in \text{Ext}_A^2 \left( \mathbb{H}^{0 \times 2 \times 2 2^i-1} \right) \]

\[ \beta_2^{-1} \beta_0^{-1} \] Novikov SS \[ \beta_{p^i-1}^{-1} \beta_0^{-1} \]

**Thm (Toda 1967)** For each \( p > 2 \)

\[ a^p_{p-1} (L_1) = \alpha, \beta_0^p \]

\[ \alpha \in \pi_{2p-1} S^0 \]

i.e. \( S^0 \), NO ANALOG at \( p = 2 \).
Lemma: There are relations
\[ \theta_j = \theta_{j-1}, \quad \beta_{p^i-1}/p^{i-1} \]
\[ \theta_1^{p_0} \theta_{j+2} = \theta_2^{p_0} \theta_{j+1}, \quad \text{for} \quad j \geq 1 \]

Also hold for \( p = 2 \)

Detection Thm. 1978: For monomial \( \theta^5 \) in
\[ \theta_j = \beta_{p^i-1}/p^{i-1}, \quad \theta^5 \quad \text{and} \quad \alpha_5 \theta^5 \]
has nontrivial image (if they exist)
in \( \pi_5 \) \( (E_{p-1} \Phi) \). Originally in terms
of \( H^*(C_p \cup \pi_x E_{p-1}) \).

These three results imply...
\[ d_2 p_{-1}(\beta^{p_{-1}/p_{-1}}) \neq 0 \text{ for all } j > 1 \]

**CAVEAT** at \( p = 3 \). The corresponding statement in the Adams SS is false:

\[ \alpha \sim (b_2) \equiv \alpha \sim l_1^3 \quad \text{dim } b_2 = 106 \]

but \( \alpha \sim l_1^3 = 0 \) in \( E_2 \)

In Novikov SS

\[ \beta_{p/9} \text{ is not a stem cycle but } \beta_{p/9} \pm \beta_7 \text{ is a permanent cycle} \]

i.e. \( \theta_1, \theta_2 \) exist at \( p = 3 \)

\( \theta_2 \) does not exist,
We do not have an equivariant approach to this problem, i.e., we do not have a $G_p$-action on $\text{MU}^{(p-1)}$ for $p > 2$. 