Monday, January 26, 2015 10:47 AM

How to define the homology and homotopy of a spectrum X.

- $H_i\,X = \lim_n H_{n+i}\,X_n.$
- $\pi_i X$ is similarly defined.

Remarks:

- X and $\tilde{X}$ have the same homotopy and homology.
- For an Ω-spectrum X, π_i X = π_i X_0 (for $i \geq 0$), but H_iX and $H_i(X_0)$ are wildly different. For the spectrum K, the mod p homology turns out to be trivial.

Why study spectra????

Short answer: Homotopy theory is nicer in the world of spectra than in the world of spaces.

If a space X is n-connected, then it behaves nicely below dimension $2n$.

Freudenthal Suspension Theorem (1937). The suspension homomorphism $\pi_{n+k}\,S^n \to \pi_{n+k+1}S^{n+1}$ is an isomorphism for $k < n-1$ and onto for $k = n-1$. This means that the group $\pi_{n+k}\,S^n$ is independent of n for $n > k+1$. It is by definition $\pi_k S^0$.

Consider a cofiber sequence, where A and X are CW-complexes with base point and f is a cellular base point preserving maps.

$$\begin{array}{l} A \\ \downarrow f \\ X \cong M\_f = \text{mapping cylinder } A \times I \cup X/(a,1)\sim f(a), (a_0 \times I)\sim point. \\ \downarrow g \\ X/A \cong M_f/(A,0) \cong C_f = \text{mapping cone of f.} \end{array}$$

There is a long exact sequence relating the homology groups of these three spaces. The homotopy groups are hard top relate to each other. If A and X are n-connected, then there is a long exact sequence of homotopy groups below dimension 2n.

A cofiber sequence as above can be extended to the right as follows.

A cofiber sequence as above can be extended to the right as follows.

$$A \xrightarrow{f} X \xrightarrow{g} X/A \xrightarrow{h} \Sigma A \xrightarrow{\Sigma f} \Sigma X \xrightarrow{\Sigma g} \Sigma X/A \xrightarrow{\Sigma h} \cdots$$

Consider the map f as above. It has a homotopy theoretic fiber (HTF) as follows. $X \cong M\_f$. We can show that A is homotopy equivlent to the space of paths in M_f that start in the subspace $A \times 0$. The HTF F is the subsapce of such paths that end at $x_0 \in X$.
There is a LES relating the homotopy groups of F, A and X.
When we have a fiber bundle

$$F \to E \xrightarrow{p} B$$

The HFT of the map p is equivalent to F. The homology groups are hard to relate.

A fiber sequence can be extended to the left
$$\ldots \to \Omega F \to \Omega A \to \Omega X \to F \to A \to X$$

In the world of spectra these two constructions coincide up to homotopy equivalence and $\Sigma\Omega X \cong X$ and $X \cong \Omega\Sigma X$. Hence we can define $\Sigma^{-1}X = \Omega X$ and $\Omega^{-1}X = \Sigma X$.

In the category of spectra (yet to be precisely defined!), fiber sequences and cofiber sequences are the same thing. We have long exact sequences in BOTH homotopy and homology.

The Serre spectral sequence is a way to compute H^*E in terms of H^*F and H^*B.

Example $E = B \times F$. If we are using field coefficients, $H^*E = H^*B \otimes H^*F$.

Consider the bigraded group $E_2^{i,j} = H^i(B; H^jF)$. If we are using field coefficients, this group is simply $H^iB \otimes H^jF$. It turns there is a homomorphism
$$d_2\colon E_2^{i,j} \to E_2^{i+2,j-1} \text{ with } d_2d_2 = 0.$$
This makes E_2 into a bigraded cochain complex. We will denote its cohomology by $E_3^{*,*}$. $E_3^{i,j} = \ker \frac{d_2}{\square} im\, d_2$.

It turns there is a homomorphism

$$d_3\colon E_3^{i,j} \to E_3^{i+3,j-2} \text{ with } d_3 d_3 = 0.$$

This makes E_3 into a bigraded cochain complex. We will denote its cohomology by $E_4^{*,*}$. $E_4^{i,j} = \ker d_2 / im\ d_2$.

It turns there is a homomorphism

$$d_r\colon E_r^{i,j} \to E_r^{i+r,j+1-r} \text{ with } d_r d_r = 0.$$

This makes E_r into a bigraded cochain complex. We will denote its cohomology by $E_{r+1}^{*,*}$. $E_{r+1}^{i,j} = \ker d_2 / im\ d_2$. For all $r > 1$.

This leads to groups $E_\infty^{i,j}$. It is a subquotient of $H^{i+j}E$. The CW-complex B has skeleta $B^0 \subset B^1 \subset B^2 \subset \cdots$. We can define subspaces of $X = E$ by $X^i = p^{-1}(B^i)$. There are maps $X^i \to X$ and hence homomorphisms $H^*X \to H^*X^i$ with kernel $F^{i+1}H^*X$. The intersection of all of these subgroups is trivial, and their union is H^*X. Then

$$E_\infty^{i,j} = F^i\, H^{i+j}X / F^{i+1}\, H^{i+j}X.$$