The Serre spectral sequence:

Let $F \to E \to B$ be a fiber bundle of CW-complexes. Suppose we know the homologies or cohomologies of $F$ and $B$, and we want to find that of $E$.

Consider the bigraded group $E_2^{i,j} = H^i(B; H^jF)$. This group vanishes if $i < 0$ or $j < 0$.

It turns out that for each bidegree there is homomorphism $d_2: E_2^{i,j} \to E_2^{i+2,j-1}$ with $d_2d_2 = 0$, so we have a bigraded cochain complex. We denote its cohomology by $E^i_3$. Inductively we have maps $d_r: E_r^{i,j} \to E_r^{i+r,j-r+1}$ with $d_rd_r = 0$, so we have a bigraded cochain complex. We denote its cohomology by $E^{i,j}_{r+1}$. Note that $E_2^{i,j} = 0$ if either $i$ or $j$ is negative. This means that for fixed $i,j$, for $r > 0$, the incoming and outgoing $d_r$ are both trivial, so $E^{i,j}_r = E^{i,j}_\infty$.

EXAMPLE

Suppose $F = S^{n-1}$. This means that $E_2^{i,0} = E_2^{i,n-1} = H^iB$ and $E_2^{i,j} = 0$ for other values of $j$. The only possible nontrivial differential is $d_n$.

Suppose our sphere bundle is the unit sphere bundle for an $n$-plane bundle $\xi$ over $B$. Then it turns out that $d_n$ is multiplication (via cup product) by a class $e(\xi) \in H^nB$ called the Euler
class of $\xi$. The kernel of $d_n$ is $E_n^{i,n-1}$ and the cokernel is $E_n^{i+n,0}$. Furthermore, there are no more differentials, so $E_{n+1} = E_\infty$. We get a long exact sequence

$$
\ldots \to H^{k-n}B \xrightarrow{d_n} H^kB \to H^kE \to H^{k-n-1}B \to \ldots
$$

Specific example. $S^1 \to S^{2n+1} \to CP^n$. Think of $S^{2n+1}$ as the unit sphere in $C^{n+1}$. The unit circle acts on it by scalar multiplication. The orbit is $CP^n$, the space of complex lines thru the origin.

This leads to $H^*E = H^*S^7$.

Consider the canonical complex line bundle $\lambda$ over $CP^\infty$. $H^*CP^\infty = Z[x]$ where $x \in H^2$. The Euler class is $x$. The fiber sequence is $S^1 \to S^\infty \to CP^\infty$.

Conclusion is that $H^*S^\infty = H^*(point)$. Consider the n-fold direct sum of $\lambda$ with itself. Then $e(n\lambda) = e(\lambda)^n = x^n$. For $n = 3$ we get $S^5 \to E \to CP^\infty$.

Conclusion: $H^*E = H^*CP^2$. Exercise: show that $E$ is homotopy equivalent to $CP^2$. 

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EXAMPLE 2. $K(Z/2,1) = F \to E \to K(Z/2,2) = B$ where $E$ is the path pace of $B$ and $F = \Omega B$. We know that $K(Z/2,1) = RP^\infty$ so its mod 2 cohomology is $Z/2[x]$ where $x \in H^1$ and $E$ has the cohomology of a point.

$$d_2(x^n x_2^i) = n x^{n-1} x_2^{i+1}.$$ 

$d_2$ is a derivation with respect to cup product, i.e., it behaves like the product rule in calculus. $d_2(x^n) = n x^{n-1} d_2(x) = n x^{n-1} x_2$. This vanishes for even $n$ but not for odd $n$. 

$$d_3((x^2)^n x_3^i) = n (x^2)^{n-1} x_3^{i+1}.$$ 

Conclusion: $H^*(K(\mathbb{Z}/2,2) = F_2[x_2, x_3, x_5, x_9, \ldots]$ where $x_{1+2^i} \in H^{1+2^i}$. 
