Varying the indexing category. Recall that $\underline{\mathcal{S}}_G$ is the TGC of functors from $\mathcal{J}_G$ to $\underline{\mathcal{T}}_G$. Within $\mathcal{J}_G$ we have the full subcategory $\mathcal{J}$ whose objects are finite dimensional vector spces with trivial G-action. Functors from $\mathcal{J}$ to $\underline{\mathcal{T}}_G$ are called *naïve* G-spectra, meaning ordinary spectra in which each space has a G-action and each structure map is equivariant. The fully faithful inclusion functor $i: \mathcal{J} \to \mathcal{J}_G$ induces a functor from $\underline{\mathcal{S}}_G$ to the category of naïve G-spectra. We will not give the latter a name because it turns out that *the two categories are equivalent*.

The formal proof involves the *Kan extension* along i, meaning a canonical way to extend a functor on $\mathcal{J}$ (such as a naive G-spectrum) to one on $\mathcal{J}_G$.

Let the category of naïve G-spectra be denoted by $\mathcal{T}_G^{\mathcal{J}}$, the category of functors $\mathcal{J} \to \mathcal{T}_G$ and the category of "genuine" G-spectra is $\mathcal{S}^G$. We have a functor $i^*: \mathcal{S}^G \to \mathcal{T}_G^{\mathcal{J}}$. We use a Kan extension along i to get a functor $i_!$ going the other way.

Informally, suppose V and W are two orthogonal G-representations of the same dimension. Then $O(V,W)$ is acted on by $O(V)$ on the right and by $O(W)$ on the left and $\mathcal{J}_G(V,W) = O(V,W)_+$. The structure map $O(V,W)_+ \wedge_{O(V)} X_V \to X_W$ is a G-equivariant homeomorphism. This means that X_W is determined by X_V, even when the G-action on V is trivial. Hence the G-spectrum X is determined by its underlying naïve G-spectrum.

The norm functor. Let $H \subseteq G$ be finite groups and let $m = |G|/|H|$. We will define a functor $N_H^G: \underline{\mathcal{S}}_H \to \underline{\mathcal{S}}_G$ that is the right adjoint of the forgetful or restriction functor $i_H^*: \underline{\mathcal{S}}_G \to \underline{\mathcal{S}}_H$. On 2/23 we saw a similar construction on the space level involving the m-fold Cartesian products. On the pointed space level we get a G-action on the m-fold smash power $Y^{(m)}$ for an H-space Y. Given a representation V of H, one has the induced representation of G,

$$ind_H^G V = \mathbf{Z}[G] \otimes_{\mathbf{Z}[H]} V.$$

It turns out that $N_H^G S^V = S^{ind_H^G V}$ and that $N_H^G S^{-V} = S^{-ind_H^G V}$.

More formally, let J be a G-set and let $\mathcal{B}_J G$ be the small category with object set J with morphisms induced by the G-action. For each $j \in J$ and $\gamma \in G$ we get a morphism $j \to \gamma(j)$. When J is a point, we abbreviate this to $\mathcal{B}G$, the one object category corresponding to G. A functor from $\mathcal{B}G$ to some category C is the same thing as an action of G on an object of C. We denote the category of such functors by $C^{\mathcal{B}G}$. Hence $\mathcal{S}^{\mathcal{B}G}$ is the category of naive G-spectra, which we have seen to be equivalent to $\underline{\mathcal{S}}_G$.

A $\mathcal{B}_J G$-diagram in $\mathcal{S}$, meaning a functor $X: \mathcal{B}_J G \to \mathcal{S}$, leads to G-actions on the spectra $\bigvee_{j \in J} X_j$ and $\bigwedge_{j \in J} X_j$, the indexed wedge and smash product of X.

For $J = G/H$ there is an inclusion $\mathcal{B}H \to \mathcal{B}_J G$ to the full subcategory containing the coset of the identity in G. It is an equivalence of categories. It has a Left Kan extension leading to an equivalence between $\mathcal{S}^{\mathcal{B}H}$ and $\mathcal{S}^{\mathcal{B}_J G}$. The indexed wedge and smash product gives us two functors from $\underline{\mathcal{S}}_H$ to $\underline{\mathcal{S}}_G$ sending an H-spectrum X to $G_+ \wedge_H X$ and $N_H^G X$.

A weak equivalence in $\mathcal{T}^G$, the category of pointed G-spaces and equivariant maps, is a map $f: X \to Y$ that induces an isomorphism $\pi_* X^H \to \pi_* Y^H$ for each $H \subseteq G$.

Recall how homotopy groups of a spectrum are defined. In the original definition, $\pi_k X = \lim_{n} \pi_{n+k} X_n$. For us it is $\pi_k^G X = \lim_{V} \pi_{V+k}^G X_V$ where $\pi_{V+k}^G Y = [S^{V+k}, Y]^G$ for a G-space Y. We also have underlying homotopy groups $\pi_k^u X = \lim_{V} \pi_{|V|+k}^u X_V$. We can define $\pi_* X^H$ for each subgroup H. A map of G-spectra $f: X \to Y$ is a weak equivalence if It induces isomorphisms $\pi_* X^H \to \pi_* Y^H$ for each subgroup H.

This is the first step toward a model structure, for which we still need to define fibrations and cofibrations.