See page 9 of Dugger's A PRIMER ON HOMOTOPY COLIMITS, for nice description of the geometric realization of a simplicial set or space. On page 12 he explains why one only needs nondegenerate simplices in $X$ in order to construct $|X|$. A nondegenerte $n$-simplex is an element of $X_n$ that is not in the image of $X_{n-1}$ under any of the degeneracy maps $s_i$. In many cases these exist only for small $n$.

**A note on homotopy colimits.** Recall the map of pushout diagrams

$$\begin{array}{ccccc} D^n & \leftarrow & S^{n-1} & \rightarrow & D^n \\ \downarrow & & \downarrow & & \downarrow \\ * & \leftarrow & S^{n-1} & \rightarrow & * \end{array}$$

in which each vertical map is an is a weak equivalence, but the pushouts or colimits are $S^n$ and * respectively. We saw that we could define a model structure on the category of pushout diagrams in which the top row is cofibrant but the bottom row is not.

Here is another approach. Replace the usual pushout of $X \leftarrow A \rightarrow Y$ by the **homotopy pushout** , which is defined to be the quotient of $X \coprod (A \times I) \coprod Y$ obtained by gluing the ends of the cylinder onto $X$ and $Y$ using the two given maps. This gives $S^{n-1}$ for both rows of the diagram above. It is called the **homotopy colimit**. There is a way to generalize it to arbitrary diagram that involves simplicial sets.

There is one other case that will concern us here. Suppose we have a diagram of the form

$$X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$$

The resulting colimit is called a **filtered colimit**. The homotopy analog is the **mapping telescope**, which is the quotient of

$$\coprod_{n \geq 0} X_n \times I$$

Where one end of the $n$th cylinder is glued onto the next one using the given map $X_n \rightarrow X_{n+1}$. It is know to be a homotopy invariant of the diagram and is denoted by $\text{holim}_{\rightarrow n} X_n$.

**Fibrant spectra.** If a spectrum $Y$ is fibrant, then any acyclic cofibration $X \rightarrow X'$ must induce a weak equivalence (of pointed $G$-spaces) $\mathcal{S}_G(X', Y) \rightarrow \mathcal{S}_G(X, Y)$. Suppose that the weak equivalence $S^{-V} \wedge S^V \rightarrow S^{-0}$ is a cofibration. Recall that for any pointed $G$-space $K$, $\mathcal{S}_G(S^{-V} \wedge K, Y) = \mathcal{T}_G(K, Y_V)$ by the Yoneda lemma. It follows that

$$\mathcal{S}_G(S^{-0}, Y) = \mathcal{T}_G(S^0, Y_0) = Y_0$$

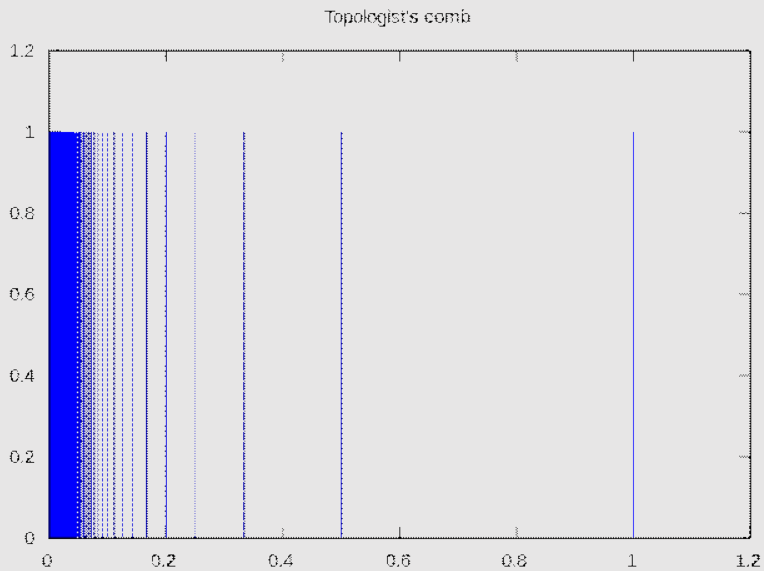
and

$$\mathcal{S}_G(S^{-V} \wedge S^V, Y) = \mathcal{T}_G(S^V, Y_V) = \Omega^V Y_V,$$

so the map $Y_0 \rightarrow \Omega^V Y_V$ is a weak equivalence for all V. Hence *fibrant spectra are what used to be called $\Omega$-spectra.* This observation predates the current definition of spectra and is due to Bousfield-Friedlander Homotopy theory of Γ-spaces, spectra, and bisimplicial sets, 1977.

**Sidebar on nondegenerate base points.** A base point $x \in X$ is **nondegenerate** if the pair $(X, \{x\})$ has the homotopy extension property (HEP), defined as follows. A pair $(X, A)$ has the **HEP** if any map $h: X \times \{0\} \cup A \times I \rightarrow Y$, called a partial homotopy, can be extended to all of $X \times I$. This condition is almost always met, meaning that counter examples tend to be pathological in nature.

Here is one. Let $X \subset I^2$ be the *comb space*, $X = \{0\} \times I \cup \bigcup_{n>0} \left\{\frac{1}{n}\right\} \times I \cup I \times \{0\}$, (illustrated below, image courtesy of [http://en.wikipedia.org/wiki/Comb_space](http://en.wikipedia.org/wiki/Comb_space)) and let $x = (0,1) \in X$. We will show the pair $(X, \{x\})$ does not have the HEP, which will mean that the base point $x$ is degenerate. Let $Y = X \times \{0\} \cup \{x\} \times I \subset X \times I$ and let $f$ be the identity map. It does not extend to $X \times I$ because its subspace $Y$ is not a retract.




There is a trick called a "adding a whisker" for dealing with degenerate base points in general. Replace the bad pair $(X, \{x\})$ with a good pair $(X', \{x'\}) \to (X, \{x\})$ constructed as follows. The space $X'$ is the union of $X$ with an interval $I$ attached to $X$ at the point $x$, and $x'$ is the other end of the interval. The map $X' \to X$ collapses $I$ to $x$. In the example above, $Y$ is the whiskered space $X'$. Adding a whisker is functorial on $(X, x)$. It serves as a left deformation for any functor that is homotopical on spaces with nondegenerate base point.