The goal of the homotopy theory we have been describing is to prove that the topological category of G-spectra $\mathcal{S}^G$ has certain properties.

Before listing them we need another category, the G-equivariant **Spanier-Whitehead category** $\mathcal{SW}^G$. Its objects are finite G-CW complexes and the set of morphisms $X \to Y$ is

$$\mathcal{SW}^G(X,Y) = \mathrm{colim}_{V_n}\{\Sigma^{V_n}X, \Sigma^{V_n}Y\},$$

where the colimit is taken over an increasing set of representations V_n chosen so that every finite dimensional orthogonal representation W is contained in some V_n, and $\{X,Y\}$ denotes the set of homotopy classes of maps $X \to Y$. The collection $\{V_n\}$ is said to be an **exhausting sequence**.

We can enlarge the category to $\mathcal{ESW}^G$ by defining additional objects $S^{-V} \wedge X$ with the property

$$\mathcal{ESW}^G(S^{-V} \wedge X, S^{-W} \wedge Y) = \mathcal{SW}^G(S^W \wedge X, S^V \wedge Y).$$

When a finite G-CW complex X is equivariantly embedded in a sphere S^V, its ***V*-dual** is defined to be the unreduced suspension of its complement. If X is V-dual to Y, then the **dual of *X*** is $DX = S^{-V} \wedge Y$.

Now here is our wish list for the homotopical category $\mathcal{S}^G$.

1. There is an adjoint pair of functors $\Sigma^\infty : \mathcal{T}^G \rightleftarrows \mathcal{S}^G : \Omega^\infty$ inducing an adjoint pair $\mathbf{L}\Sigma^\infty : \mathrm{Ho}(\mathcal{T}^G) \rightleftarrows \mathrm{Ho}(\mathcal{S}^G) : \mathbf{R}\Omega^\infty$ on homotopy categories.

   The suspension spectrum functor is defined by $(\Sigma^\infty K)_V = S^V \wedge K$ for a pointed G-space K, and the 0-space functor is defined by $\Omega^\infty Y = Y_0$ for a spectrum Y. Recall that a map of spectra $f: X \to Y$ is a collection of maps $f_V: X_V \to Y_V$ that respecting the structure maps for X and Y. When $X = \Sigma^\infty K$, each f_V is determined by f_0. It follows that the two functors are adjoint.

2. The symmetric monoidal structure on $\mathcal{S}^G$ induces a closed symmetric monoidal structure on the homotopy category $\mathrm{Ho}(\mathcal{S}^G)$ and the functor $\mathbf{L}\Sigma^\infty$ is symmetric monoidal.

3. The functor $\mathbf{L}\Sigma^\infty$ extends to a fully faithful, symmetric monoidal embedding of $\mathcal{SW}^G$ into $\mathrm{Ho}(\mathcal{S}^G)$.

4. The objects S^V are invertible in $\mathrm{Ho}(\mathcal{S}^G)$ under the smash product, so the above embedding of $\mathcal{SW}^G$ extends to an embedding of $\mathcal{ESW}^G$.

5. Arbitrary coproducts (denoted ∨) exist in $\mathrm{Ho}(\mathcal{S}^G)$ and can be computed by the formation of wedges. If $\{X_\alpha\}$ is a collection of objects of $\mathcal{S}^G$ and K is a finite G-CW complex, then the map

$$\bigoplus_{\alpha} Ho(\mathcal{S}^G (K, X_\alpha) \to \mathrm{Ho}(\mathcal{S}^G)(K, \bigvee_{\alpha} X_\alpha)$$

is an isomorphism.

6. Up to weak equivalence every object X is presentable in $\mathcal{S}^G$ in as a homotopy colimit of

$$\to \cdots \to S^{-V_n} \wedge X_{V_n} \to S^{-V_{n+1}} \wedge X_{V_{n+1}} \to \cdots$$

with the V_n as above and each X_{V_n} is weakly equivalent to a G-CW complex.

To make this work we need to deal with indexed wedges, smash products, symmetric powers and their compositions. The word "indexed" here means that indexing sets of wedges and smash products may have a G-action that needs to be taken into account. For example the left and right adjoints of the forgetful functor $\mathcal{S}^G \to \mathcal{S}^H$ are the wedge and smash product indexed by the G-set G/H, the right adjoint being the norm functor N_H^G. This technicality is discussed extensively in A.3 of [HHR], and there are no surprises. As indicated before, there *are* surprises with symmetric product functor.

These functors are not homotopical in general. We need a subcategory of $\mathcal{S}^G$ on which they are. Eventually this will lead us to a class of spectra that are the cofibrant objects in a cofibrantly generated model structure on $\mathcal{S}^G$. Recall the definition (4/14) of a left deformable functor $F: \mathcal{C} \to \mathcal{D}$ between homotopical categories. There is a dual notion of right deformable functor. When we have an adjoint pair

$$F: \mathcal{C} \rightleftarrows \mathcal{D}: G$$

with F left deformable and G right deformable, then the total derived functors (reverse handed Kan extensions) form another adjoint pair

$$\mathbf{L}F: \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}): \mathbf{R}G,$$

such as in **1** above.

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Definition. Let $\mathcal{C}$ by a complete topological category. (This means that products of its objects with topological spaces are defined. $\mathcal{S}^G$ is such a category.) A morphism $i : A \to X$ in $\mathcal{C}$ is an ***h*-cofibration** if it has the homotopy extension property: given $f : X \to Y$ and a homotopy $h : A \otimes [0,1] \to Y$ with $h| A \otimes \{0\} = f \circ i$ there is an extension of *h* to $H : X \otimes [0,1] \to Y$.

It turns out that *h*-cofibrations in $\mathcal{S}^G$ are objectwise closed inclusions, i.e. a map $f: X \to Y$ is an h-cofibration if each map $f_V: X_V \to Y_V$ is a closed inclusion.

Definition. A functor $F: \mathcal{C} \to \mathcal{D}$ between homotopical categories is **flat** if it is homotopical and preserves colimits. A morphism $f : A \to X$ in $\mathcal{C}$ is **flat** if for every morphism $A \to B$ and every weak equivalence $B \to B'$, the map $X \cup_A B \to X \cup_A B'$ is a weak equivalence.

In other words a morphism f is flat if and only if "cobase change along f" preserves weak equivalences. Since cobase change is a left adjoint this is equivalent to the flatness of the cobase change functor.

It can be shown that a weak equivalence of pushout diagrams in which the maps are flat induces a weak equivalence of pushout objects, even if only one of the morphisms in the target diagram is flat.

Now suppose that $(\mathcal{C}, \otimes, 1)$ is a closed symmetric monoidal category, equipped with a class $\mathcal{W}$ of weak equivalences, making $\mathcal{C}$ into a homotopical category.

Definition. An object $X \in \mathcal{C}$ is **flat** if the functor $X \otimes (-)$ is flat.

When every object admits a weak equivalence from a flat one, then tensoring any object with a weak equivalence of flat objects gives another weak equivalence of flat objects.

Showing that a symmetric monoidal structure on $\mathcal{C}$ induces one on $\mathrm{Ho}(\mathcal{C})$ essentially comes down to exhibiting enough flat objects in $\mathcal{C}$.

Basic homotopy theory in $\mathcal{S}^G$. Here are some properties easily proved in Appendix B.3 of [HHR].

- Filtered colimits (also known as direct limits) along closed inclusions (i.e. h-cofibrations) are homotopical.
- Let F be the homotopy fiber of a map of spectra $f : X \to Y$, namely the pullback along f of the map $PY \to Y$, where PY is the path spectrum of Y. Then we get the expected long exact sequence of equivariant homotopy groups for F, X and Y.
- The suspension homomorphism of equivariant homotopy groups induced by smashing with S^1 is an isomorphism.
- Let $X \to Y$ be an h-cofibration. Then its mapping cylinder $CX \cup Y$ is weakly equivalent to the quotient Y/X and we get the expected long exact sequence of equivariant homotopy groups for X, Y and Y/X with the connecting homomorphism induced by the map $CX \cup Y \to \Sigma X$.
- The h-cofibrations in $\mathcal{S}^G$ are flat.
- Infinite wedges and finite products are homotopical, and a finite wedge is weakly equivalent to the corresponding finite product.
- The category $\mathrm{Ho}(\mathcal{S}^G)$ is additive, and admits finite products and arbitrary coproducts. The coproducts are given by wedges and the finite products by finite products.

The suspension functor $\Sigma^\infty : \mathcal{T}^G \to \mathcal{S}^G$ is homotopical on spaces with nondegenerate base point, because the functor given by smashing with a fixed point space is homotopical on such pointed spaces. The whisker trick above allows us to define a left deformation on $\mathcal{T}^G$, replacing each pointed space (X, x) with the corresponding "whiskered" space (X', x').

The zero-space functor Ω^∞ sending a spectrum Y to the space Y_0 is far from homotopical since every spectrum is weakly equivalent to one whose zeroth space is a point. The total right derived functor of $\Omega^\infty\colon \mathcal{S}^G \to \mathcal{T}^G$ is $\mathbf{R}\Omega^\infty Y = \mathrm{holim}_\to \Omega^{V_n} Y_{V_n}$, for an exhaustive sequence $\{V_n\}$, where $\mathrm{holim}_\to$ denotes the mapping telescope of the indicated filtered colimit, which is known to be homotopy invariant. To see that it is $\mathbf{R}\Omega^\infty Y$, note that there is a weak equivalence $Y \to Y'$ with $\Omega^\infty Y' = \mathrm{holim}_\to \Omega^{V_n} Y_{V_n}$. The target is defined by $Y'_V = \mathrm{holim}_\to \Omega^{V_n} Y_{V \oplus V_n}$. In some classical literature this has been called the Ω-spectrum associated with Y, or the spectrum associated with the prespectrum Y. It turns out that once we have defined a model structure on $\mathcal{S}^G$, *Y' is the fibrant replacement of Y*.

Definition (B.57 of [HHR]). An equivariant orthogonal spectrum is **cellular** if it is in the smallest subcategory of $\mathcal{S}^G$ containing the spectra of the form $G_+ \wedge_H S^{-V} \wedge S^k_+$ with V a representation of H and $k \geq 0$ and which is closed under the formation of arbitrary coproducts, the formation of mapping cones, and the formation of filtered colimits along h-cofibrations.

The small object argument shows that every X receives, functorially, a weak equivalence $\widetilde{X} \to X$ from a cellular $\tilde{X}$. We will see that cofibrant spectra are all cellular but not the converse. Cellular spectra can be shown to be flat so smashing with them is homotopical.

Spectra as a model category. The **positive complete model structure** on $\mathcal{S}^G$ can be defined in terms of the following set of cofibrations.

$$\mathcal{A}_{\mathrm{cof}} = \{G_+ \wedge_H S^{-V} \wedge S^{n-1}_+ \to G_+ \wedge_H S^{-V} \wedge D^n_+ : n \geq 0, H \subset G, V^H \neq 0\}$$

We define the class $\mathcal{S}^G_{\mathrm{cof}} \subset \mathcal{S}^G$ of **positive complete cofibrations** to be the smallest collection of maps in $\mathcal{S}^G$ containing the maps in $\mathcal{A}_{\mathrm{cof}}$ and which is closed under coproducts, cobase change along arbitrary maps, and filtered colimits.

A **positive complete fibration** (or just fibration) is a map having the right lifting property with respect to the class of maps in $\mathcal{S}^G_{\mathrm{cof}}$ which are stable weak equivalences. For this it is helpful to have a generating set $\mathcal{B}_{\mathrm{acylic}}$ of acyclic cofibrations. As one would expect, it includes the set

$$\{G_+ \wedge_H S^{-V} \wedge D^n_+ \to G_+ \wedge_H S^{-V} \wedge I \times D^n_+ : n \geq 0, H \subset G, V^H \neq 0\},$$

but that is not all. We also need *corner maps* (to be defined below) for the smash products of the inclusions $S^{n-1}_+ \to D^n_+$ for $n > 0$ with the weak equivalences

$$G_+ \wedge_H \left(S^{-V\oplus W} \wedge S^W\right) \to G_+ \wedge_H \tilde{S}^{-V}$$

where $V^H \neq 0$ and W is arbitrary. Here $\tilde{S}^{-V}$ is obtained from either factorization (they are the same since the composite is a weak equivalance) of

$$S^{-V\oplus W} \wedge S^W \to \tilde{S}^{-V} \to S^{-V}$$

in $\mathcal{S}^H$ given by the small object argument.

What is a **corner map**? Given a map $A \to B$ and a weak equivalence $C \to D$, we get a diagram

$$A \wedge C \to B \wedge C$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ A \wedge D & \rightarrow & B \wedge D \end{array}$$

in which the vertical maps are weak equivalences. The corner map is the unique $g: P \rightarrow B \wedge D$ from the pushout P of the top and left maps. It is a weak equivalence but not the identity in general.

The condition $V^H \neq 0$ is the *positivity condition*. It is needed for the study of commutative algebras. It means that the sphere spectrum S^{-0} is not cofibrant, but $S^{-1} \wedge S^1$ is.

All cofibrations defined as a above are h-cofibrations and hence flat. All cofibrant oblects are cellular and hence flat.