Here is a simple example of a functor that fails to preserve homotopy equivalences. It is taken from a very helpful introduction to model categories by Dwyer and Spalinski.

Let \( D \) denote the category \( \{a \leftarrow b \rightarrow c\} \), \( \textbf{Top} \) the category of topological spaces, and \( \textbf{Top}^D \) the category of functors \( D \to \textbf{Top} \), i.e., pushout diagrams in \( \textbf{Top} \). Then we have the functor colim: \( \textbf{Top}^D \to \textbf{Top} \) which assigns to each diagram its pushout. It is left adjoint to the functor \( \Delta: \textbf{Top} \to \textbf{Top}^D \) which assigns to each pointed space \( X \) the constant \( X \)-valued diagram. A morphism in \( \mathcal{T}^D \) is the obvious sort of commutative diagram. Consider the morphism
\[
\begin{array}{ccc}
D^n & \to & D^n \\
\downarrow & & \downarrow \\
\ast & \to & S^{n-1}
\end{array}
\]
in which each vertical map, and hence the morphism in \( \textbf{Top}^D \), is a weak equivalence. However the pushout of the top row (where the two maps are inclusion of the boundary) is \( S^n \), while that of the bottom row is a point. Thus the pushout functor fails to preserve this weak equivalence.

It turns out there is a model structure on \( \textbf{Top}^D \) in which the top row is cofibrant but the bottom row is not, and the pushout functor DOES preserve weak equivalences between cofibrant objects. Let \( f: X \to Y \) be a morphism in \( \textbf{Top}^D \). It consists of three maps \( f_a: X_a \to Y_a \), \( f_b: X_b \to Y_b \) and \( f_c: X_c \to Y_c \).

We define the model structure by saying that \( f \) is a weak equivalence/fibration if each of the three maps is, but the definition of a cofibration is more complicated. Let \( \partial_b(f) = X_b \) and define \( \partial_a(f) \) to be the pushout of
\[
\begin{array}{ccc}
X_b & \to & X_a \\
\downarrow & & \downarrow \\
Y_b & \to & \partial_a(f)
\end{array}
\]
with a similar definition for \( \partial_c(f) \). For each index we get a map \( i_*(f): \partial_*(f) \to Y_* \). We say that \( f \) is a cofibration if each of these three maps is. It is a routine exercise (Dwyer-Spalinski Prop.10.6) to verify that this defines a model category structure on \( \textbf{Top}^D \).

An object \( X \) is cofibrant iff \( X_b \) is a CW-complex and the two maps from it are cofibrations. In the example above, the top row is cofibrant but the bottom row is not.

Given a small category \( J \) and a model category \( \mathcal{C} \), it is not generally clear how to define a model structure on the diagram category \( \mathcal{C}^J \). The case of greatest interest to us is \( \mathcal{T}_G^{J_G} \), the category of \( G \)-spectra.