

Before proceeding further, consider the following disturbing example. For a G -spectrum X , let $\text{Sym}^k X = X^{(k)}/\Sigma_k$, the k -fold symmetric smash product. Thus we get a functor

$$\text{Sym}^k: \mathcal{S}^G \rightarrow \mathcal{S}^G$$

Since S^0 is the unit for smash product, we have $\text{Sym}^k S^{-0} = S^{-0}$.

Now consider the weak equivalence $S^{-1} \wedge S^1 \rightarrow S^{-0}$. The n th space of the spectrum S^{-0} is S^n , while the one for $S^{-1} \wedge S^1$ is $\mathcal{J}_G(1, n) \wedge S^1$. $\mathcal{J}_G(1, n)$ is the tangent Thom space of S^{n-1} . And for $\mathcal{J}_G(1, n) \wedge S^1$, the n th space is the Thom space for the trivial n -plane bundle over S^{n-1} . It follows that $(\mathcal{J}_G(1, n) \wedge S^1)^{(k)}$ is the Thom space for the trival kn -plane bundle over $S^{k(n-1)}$ with the symmetric group Σ_k acting on the base, whose orbit space we denote by $X_{k,n}$. The connectivity of $S^{k(n-1)}$ means that $X_{k,n}$ is equivalent to the classifying space $B\Sigma_k$ through a range of dimensions that increases with n .

This implies that $\text{Sym}^k(S^{-1} \wedge S^1)$ is equivalent to the suspension spectrum of the space $B\Sigma_k$. Hence the functor Sym^k does *not* convert the weak equivalence $S^{-1} \wedge S^1 \rightarrow S^0$ to a weak equivalence. This is bad news for doing homotopy theory in the category of spectra.

Such difficulties can arise in the homotopy theory of spaces as well. Classically we avoid them by limiting our attention to CW-complexes. In model category language, these are the cofibrant objects in the category of topological spaces. They are typically determined by a set of generating cofibrations, as AL has explained. The set often chosen is

$$\{S^{n-1} \rightarrow D^n: n \geq 0\}.$$

For G -spaces the collection we want is

$$\{G_+ \wedge_H S^{n-1} \rightarrow G_+ \wedge_H D^n: n \geq 0, H \subseteq G\}.$$

This leads to the definition of a G -CW-complex.

What to do for the category \mathcal{S}^G ? One is tempted to use the set

$$\{G_+ \wedge_H S^{-V} \wedge S_+^{n-1} \rightarrow G_+ \wedge_H S^{-V} \wedge D_+^n: n \geq 0, H \subseteq G, V \text{ is orthogonal rep of } H\},$$

where V ranges over all representations of the subgroup H . This will not do, because it would make both $S^{-1} \wedge S^1$ and S^0 cofibrant. We need a model structure in which the functor Sym^k takes a weak equivalence between cofibrant objects to a weak equivalence. The example above shows that $S^{-1} \wedge S^1$ and S^0 cannot both be cofibrant.

It turns out that the fix is to put an additional requirement on V , namely that it has a nonzero H -invariant vector. This is called the *positivity condition*. In the nonequivariant case (meaning G is trivial) this means $\dim V > 0$. Under it $S^{-1} \wedge S^1$ is cofibrant, but S^0 is not. The resulting model structure on \mathcal{S}^G is called the *positive complete model structure*. We will see later that it has other technical advantages in addition to the good behavior of Sym^k .