Inside the proof of the Kervaire invariant theorem
or
How I got bitten by the equivariant bug

Math 549

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Prelude

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Chart by Dan Isaksen
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Drawing by Bob Bruner
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Our strategy is to construct a nonconnective ring spectrum \( \Omega \) having a unit map \( S^0 \to \Omega \) with the following properties.

(i) Detection Theorem. It has an Adams-Novikov spectral sequence (which is a device for calculating homotopy groups) in which the image of each \( \theta_j \) is nontrivial. This means that if \( \theta_j \) exists, we will see its image in \( \pi_\ast(\Omega) \).

(ii) Periodicity Theorem. It is 256-periodic, meaning that \( \pi_k(\Omega) \) depends only on the reduction of \( k \) modulo 256.

(iii) Gap Theorem. \( \pi_k(\Omega) = 0 \) for \( -4 < k < 0 \). This property is our zinger. Its proof involves a new tool we call the slice spectral sequence.
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The argument for $\theta_j$ for larger $j$ is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \mod 256$ for $j \geq 7$. 
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Real cobordism

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We denote the resulting $C_2$-spectrum by $MU_R$. 
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The $C_2$-spectrum $MU_R$ has been studied extensively.
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Constructing our spectrum $\Omega$

For $G = C_8$, we can form the norm $N^{G}_{C_2} MU_R$, which we abbreviate by $MU^{((G))}$. 
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Calculations show that there is an element $\Delta \in \pi^G_{256} D^{-1} \mathcal{M}U^{((G))}$ such that the induced map

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For $G = C_8$, we can form the norm $N_{C_2}^G MU_R$, which we abbreviate by $MU^{((G))}$. It is underlain by the 4-fold smash power $MU^\wedge 4$ with the group $G$ permuting the $C_2$-invariant factors.

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is an equivariant homotopy equivalence. Our $\Omega$ is the $G$-fixed point spectrum of $D^{-1} MU^{((G))}$. 
The slice spectral sequence

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Again, $P_n^S$, the category of $(n - 1)$-connected spectra, is generated by the set

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$G^+ \wedge S^m$ and $S^m_{\rho}$.

Here $G^+ \wedge S^m$ is the wedge of two $m$-spheres that are interchanged by the generator $\gamma \in C_2$.

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We will call these spectra slice spheres.
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Let $S^G$ denote the category of $G$-spectra.
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Let \( S^G \) denote the category of \( G \)-spectra. Define \( P_n S^G \) to be the subcategory generated by the elements of \( T_n^G \), i.e., by slice spheres of dimension \( \geq n \).

This filtration of \( S^G \) leads to the slice spectral sequence. Unlike the classical Postnikov spectral sequence, it is extremely useful. It maps to the classical one under the forgetful functor \( S^G \rightarrow S \).
The slice spectral sequence for $G = C_2$ (continued)

For $G = C_2$ the generalization of

$$T_n = \{ S^m : m \geq n \}$$

is

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Let $S^G$ denote the category of $G$-spectra. Define $P_nS^G$ to be the subcategory generated by the elements of $T^n_G$, i.e., by slice spheres of dimension $\geq n$.

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Let $S^G$ denote the category of $G$-spectra. Define $P_nS^G$ to be the subcategory generated by the elements of $T_n^G$, i.e., by slice spheres of dimension $\geq n$.

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The slice spectral sequence for general groups $G$

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The definitions above can be generalized to an arbitrary finite group $G$. For each subgroup $H \subseteq G$ and each integer $m$, we define $G^+ \wedge H S^m \rho_H$ to be a slice sphere of dimension $m|H|$, where $\rho_H$ is the regular representation. Then we define $T_G n = \{ G^+ \wedge H S^m \rho_H : m|H| \geq n, H \subseteq G \}$, the set of slice spheres of dimension $\geq n$. 

The case $G = C_2$

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Determining the slices of a $G$-spectrum $X$ is not easy in general.
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where $W_n$ is a wedge of $n$-dimensional slice spheres and $\mathbb{H}Z$ is the integer Eilenberg-Mac Lane spectrum with trivial $G$-action. $W_n$ never has a wedge summand of the form $G_+ \wedge S^n$. 


The slice spectral sequence for $\text{MU}_R$

We have a complete description of the slice spectral sequence for $\text{MU}_R$. 

Prelude
- Browder's theorem
- The Adams spectral sequence
- The Mahowald Uncertainty Principle
- Differentials

The HHR strategy
- The spectrum $\Omega$

Equivariant stable homotopy theory
- Two useful functors
- Representation spheres
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- Constructing our spectrum $\Omega$

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The spectrum $\Omega$ is the fixed point spectrum for a $G$-spectrum $D^{-1}MU^{((G))}$, where $G = C_8$. 
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The spectrum $\Omega$ is the fixed point spectrum for a $G$-spectrum $D^{-1}MU^{((G))}$, where $G = C_8$.

The homotopy of $D^{-1}MU^{((G))}$ and its fixed point spectra can be studied with the slice spectral sequence.
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The homotopy of $D^{-1}MU^{((G))}$ and its fixed point spectra can be studied with the slice spectral sequence. Its input is the homotopy groups of wedges of spectra of the form

$$K_{m,H} = G_+ \wedge_{H} H S^{mH} \wedge H\mathbb{Z}$$

for integers $m$ and nontrivial subgroups $H \subseteq G$. 
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for integers $m$ and nontrivial subgroups $H \subseteq G$. This means that its $G$-fixed point spectrum $\Omega$ is built out of copies of $K_{m,H}^G$, the $G$-fixed point spectrum of $K_{m,H}$.
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We will show that $\pi_{-2}K_{m,H}^G$ vanishes in every case.
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$\pi_{-2}\Omega$ never had a chance!
The proof of the Gap Theorem (continued)

How do we compute $\pi_* K^G_{m,H}$?
The proof of the Gap Theorem (continued)

How do we compute $\pi_* K^G_{m,H}$? We begin with the underlying homotopy groups of $K_{m,H}$ for $m \geq 0$. 

How I got bitten
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The proof of the Gap Theorem (continued)

How do we compute $\pi_*^G K_{m,H}^G$? We begin with the underlying homotopy groups of $K_{m,H}$ for $m \geq 0$. We have

$$\pi_*^u K_{m,H} = \pi_*^u G_+ \wedge_{H} S^{m \rho_H} \wedge H\mathbb{Z}$$

$$= H_*^u G_+ \wedge_{H} S^{m \rho_H} \quad \text{(underlying homology)}$$

$$= \bigoplus_{|G/H|} H_* S^{|H|}.$$
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$G_+ \wedge_H S^{m \rho H}$ is a finite $G$-CW complex. This means that it has a reduced cellular chain complex $C_{m,H}^*$ of $\mathbb{Z}[G]$-modules.
The proof of the Gap Theorem (continued)

How do we compute $\pi_* K_{m,H}^G$? We begin with the underlying homotopy groups of $K_{m,H}$ for $m \geq 0$. We have

$$\pi_* K_{m,H} = \pi_*^G \mathbb{P}^{m \rho_H} \wedge H\mathbb{Z}$$

$$= H_*^{G+} \wedge H \mathbb{Z}$$

(underlying homology)

$$= \bigoplus_{|G/H|} H_* S^m H$$

$G+ \wedge H \mathbb{Z}$ is a finite $G$-CW complex. This means that it has a reduced cellular chain complex $C_{*,H}^m$ of $\mathbb{Z}[G]$-modules. Describing it is a geometric exercise.
The proof of the Gap Theorem (continued)

How do we compute $\pi_* K^G_{m,H}$? We begin with the underlying homotopy groups of $K_{m,H}$ for $m \geq 0$. We have

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$$= H_* G_+ \wedge_H S^{mpH} \quad \text{(underlying homology)}$$

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$G_+ \wedge_H S^{mpH}$ is a finite $G$-CW complex. This means that it has a reduced cellular chain complex $C_{*,H}^m$ of $\mathbb{Z}[G]$-modules. Describing it is a geometric exercise.

For $G_+ \wedge_H S^{-mpH}$, we can use the $\mathbb{Z}$-linear dual of $C_{*,H}^m$, 

π∗Km,H = π∗G+ ∧H SmρH ∧HZ

= H∗ G+ ∧H SmρH (underlying homology)

= ⊕ |G/H| H∗ Sm|H|.

G+ ∧H SmρH is a finite G-CW complex. This means that it has a reduced cellular chain complex C∗,Hm of Z[G]-modules. Describing it is a geometric exercise.
The proof of the Gap Theorem (continued)

How do we compute $\pi_* K_{m,H}^G$? We begin with the underlying homotopy groups of $K_{m,H}$ for $m \geq 0$. We have

$$\pi_* K_{m,H} = \pi_* G_+ \wedge_H S^{m\rho_H} \wedge H\mathbb{Z}$$

$$= H_* G_+ \wedge_H S^{m\rho_H} \quad \text{(underlying homology)}$$

$$= \bigoplus_{|G/H|} H_* S^m|H|.$$ 

$G_+ \wedge_H S^{m\rho_H}$ is a finite $G$-CW complex. This means that it has a reduced cellular chain complex $C^{m,H}_*$ of $\mathbb{Z}[G]$-modules. Describing it is a geometric exercise.

For $G_+ \wedge_H S^{-m\rho_H}$, we can use the $\mathbb{Z}$-linear dual of $C^{m,H}_*$, which we denote by $C^{-m,H}_*$. 

The proof of the Gap Theorem (continued)

It follows that

$$\pi_* K^G_{m,H} = H_* \left( (C^{m,H})^G \right)$$

for all $m$ and $H$.

We now analyze $C^{m,H}$ and $(C^{m,H})^G$ for $m \geq 0$. 
The proof of the Gap Theorem (continued)

It follows that

$$\pi_\ast K^G_{m,H} = H_\ast \left( (C^{m,H})^G \right) \quad \text{for all } m \text{ and } H.$$ 

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The proof of the Gap Theorem (continued)

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**WARNING** Fixed points do not commute with smash products,
The proof of the Gap Theorem (continued)

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The proof of the Gap Theorem (continued)

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\begin{verbatim}
WARNING Fixed points do not commute with smash products, so $(G_+ \wedge H^{m\rho_H} \wedge HZ)^G$ is not the same as $(G_+ \wedge H^{m\rho_H})^G \wedge HZ$, and $H_* \left( (C^{m,H})^G \right)$ is not the homology of $(G_+ \wedge H^{m\rho_H})^G = \left\{ \begin{array}{ll} S^m & \text{for } H = G \\ * & \text{otherwise.} \end{array} \right.$
\end{verbatim}
We are analyzing $C^{m,H}$ and $(C^{m,H})^G$ for $m \geq 0$. 
We are analyzing $C^m,^H$ and $(C^m,^H)^G$ for $m \geq 0$. The bottom $G$-cell of $G_+ \wedge_H S^{m\rho_H}$ is

$$(G_+ \wedge_H S^{m\rho_H})^H = G_+ \wedge_H S^m$$

in dimension $m$,
The proof of the Gap Theorem (continued)

We are analyzing $C^{m,H}$ and $(C^{m,H})^G$ for $m \geq 0$. The bottom $G$-cell of $G_+ \smash[b]{\wedge_{H} S^{m\rho_H}}$ is

$$(G_+ \smash[b]{\wedge_{H} S^{m\rho_H}})^H = G_+ \smash[b]{\wedge_{H} S^{m}}$$

in dimension $m$, while the top cell is in dimension $m|H|$. Similar statements hold for $C_{-m,H}$, $H$, $C_{-m-1,H}$, and their fixed point subcomplexes.
The proof of the Gap Theorem (continued)

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in dimension $m$, while the top cell is in dimension $m |H|$. Similar statements hold for $C^{m,H}$, $C^{-m,H}$ and their fixed point subcomplexes.
The proof of the Gap Theorem (continued)

The bottom $G$-cell of $G_+ \wedge_H S^{m\rho_H}$ is

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It follows that for $m \geq 0$, $\pi_i K^G_{m,H}$ is trivial unless $m \leq i \leq m|H|$. 
The proof of the Gap Theorem (continued)

The bottom $G$-cell of $G_+ \wedge_H S^{m\rho_H}$ is

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in dimension $m$, while the top cell is in dimension $m|H|$. Similar statements hold for $C^{m,H}$, $C^{-m,H}$ and their fixed point subcomplexes.

It follows that for $m \geq 0$, $\pi_i K_{m,H}^G$ is trivial unless $m \leq i \leq m|H|$, and $\pi_i K_{-m,H}^G$ is trivial unless $-m \geq i \geq -m|H|$.
The proof of the Gap Theorem (continued)

The bottom $G$-cell of $G_+ \wedge_{H} S^{m\rho_H}$ is

$$(G_+ \wedge_{H} S^{m\rho_H})^H = G_+ \wedge_{H} S^m$$

in dimension $m$, while the top cell is in dimension $m|H|$. Similar statements hold for $C^{m,H}$, $C^{-m,H}$ and their fixed point subcomplexes.

It follows that for $m \geq 0$, $\pi_i K^G_{m,H}$ is trivial unless $m \leq i \leq m|H|$, and $\pi_i K^G_{-m,H}$ is trivial unless $-m \geq i \geq -m|H|$.

For the Gap Theorem we want to show that $\pi_{-2} K^{G}_{m,H} = 0$ in all cases.
The proof of the Gap Theorem (continued)

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For the Gap Theorem we want to show that $\pi_{-2} K_{m,H}^G = 0$ in all cases. From the above we see that the only values of $m$ we need to consider are $m = -1$ and $m = -2$. 
The proof of the Gap Theorem (continued)

For the Gap Theorem we want to show that $\pi_{-2} K^G_{m, H} = 0$ in all cases, and the only values of $m$ we need to consider are $m = -1$ and $m = -2$. 
The proof of the Gap Theorem (continued)

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The proof of the Gap Theorem (continued)

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For simplicity I will do this for $H = G = C_2$, this being similar in essence to the cases where $G = C_8$.

For $m = 1$, $C^1,C_2$ is the reduced $C_2$-cellular chain complex for $S^{\rho_2}$. It is

$$
\begin{array}{ccc}
1 & 2 \\
\mathbb{Z} & \nabla & \mathbb{Z}[C_2]
\end{array}
$$

where $\nabla$ is the augmentation map sending the generator $\gamma$ to 1.
The proof of the Gap Theorem (continued)

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\[
\begin{array}{c}
1 \\
\mathbb{Z} \\
\end{array} \xleftarrow{\nabla} \begin{array}{c}
2 \\
\mathbb{Z}[C_2] \\
\end{array}
\]

where $\nabla$ is the augmentation map sending the generator $\gamma$ to 1.

Its $\mathbb{Z}$-linear dual $C^{-1}, C_2$ is

\[
\begin{array}{c}
-1 \\
\mathbb{Z} \\
\end{array} \xrightarrow{\Delta} \begin{array}{c}
-2 \\
\mathbb{Z}[C_2] \\
\end{array}
\]

where $\Delta$ is the diagonal embedding sending 1 to $1 + \gamma$. 
The proof of the Gap Theorem (continued)

$C^{-1, C_2}$ is

\[
\begin{array}{ccc}
-1 & & -2 \\
\mathbb{Z} & \xrightarrow{\Delta} & \mathbb{Z}[C_2]
\end{array}
\]

where $\Delta$ is the diagonal embedding sending 1 to $1 + \gamma$.
The proof of the Gap Theorem (continued)

$C^{-1, C_2}$ is

$$
\begin{pmatrix}
-1 & -2 \\
\end{pmatrix}
$$

\[ Z \xrightarrow{\Delta} Z[C_2] \]

where $\Delta$ is the diagonal embedding sending 1 to $1 + \gamma$.

Passing to fixed points gives

$$
\begin{pmatrix}
-1 & -2 \\
\end{pmatrix}
$$

\[ Z \xrightarrow{1} Z \]
The proof of the Gap Theorem (continued)

\[ C_{-1, C_2} \]

\[
\begin{array}{c}
\mathbb{Z} \\
\mathbb{Z}^2 \end{array}
\xrightarrow{\Delta}
\begin{array}{c}
\mathbb{Z}[C_2] \\
\mathbb{Z} \end{array}
\]

where \( \Delta \) is the diagonal embedding sending 1 to \( 1 + \gamma \).

Passing to fixed points gives

\[
\begin{array}{c}
\mathbb{Z} \\
\mathbb{Z} \end{array}
\xrightarrow{1}
\begin{array}{c}
\mathbb{Z} \\
\mathbb{Z} \\
\mathbb{Z} \end{array}
\]

This has trivial homology, so \( \pi_{-2} K_{C_2}^{C_2} = 0 \).
The proof of the Gap Theorem (continued)

Now we have to deal with $m = -2$. 
The proof of the Gap Theorem (continued)

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$C^{-2, C_2}$ is

$$
\begin{array}{ccc}
-2 & -3 & -4 \\
\mathbb{Z} & \xrightarrow{\Delta} & \mathbb{Z}[C_2] \xrightarrow{1-\gamma} \mathbb{Z}[C_2]
\end{array}
$$

Passing to fixed points gives

$$
\begin{array}{ccc}
-2 & -3 & -4 \\
\mathbb{Z} & \xrightarrow{\Delta} & \mathbb{Z}[C_2] \xrightarrow{1-\gamma} \mathbb{Z}[C_2]
\end{array}
$$

This has nontrivial homology, but only in dimension $-4$, so again $\pi_{-2} K C_2 = 0$.

This completes the proof of the Gap Theorem.
The proof of the Gap Theorem (continued)

Now we have to deal with $m = -2$.

$C^{-2, C_2}$ is

$$
\begin{array}{ccc}
-2 & -3 & -4 \\
\mathbb{Z} & \rightarrow & \mathbb{Z}[C_2] \\
\Delta & & 1-\gamma \\
\rightarrow & & \rightarrow \\
\mathbb{Z}[C_2] & & 
\end{array}
$$

Passing to fixed points gives

$$
\begin{array}{ccc}
-2 & -3 & -4 \\
\mathbb{Z} & \rightarrow & \mathbb{Z} \\
1 & & 0 \\
\rightarrow & & \rightarrow \\
\mathbb{Z} & & 
\end{array}
$$

Passing to fixed points gives

$$
\begin{array}{ccc}
-2 & -3 & -4 \\
\mathbb{Z} & \rightarrow & \mathbb{Z} \\
1 & & 0 \\
\rightarrow & & \rightarrow \\
\mathbb{Z} & & 
\end{array}
$$
The proof of the Gap Theorem (continued)

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-2 & -3 & -4 \\
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\end{array}
\]

Passing to fixed points gives

\[
\begin{array}{ccc}
-2 & -3 & -4 \\
\mathbb{Z} & 1 & \mathbb{Z} & 0 & \mathbb{Z}
\end{array}
\]

This has nontrivial homology, but only in dimension $-4$, so again $\pi_{-2}K_{-2,C_2} = 0$. 

1.34

The proof of the Gap Theorem (continued)

Now we have to deal with $m = -2$.

$C^{-2,C_2}$ is

\[
\begin{array}{ccc}
-2 & -3 & -4 \\
\mathbb{Z} & \Delta & \mathbb{Z}[C_2] & 1-\gamma & \mathbb{Z}[C_2]
\end{array}
\]

Passing to fixed points gives

\[
\begin{array}{ccc}
-2 & -3 & -4 \\
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\]

This has nontrivial homology, but only in dimension $-4$, so again $\pi_{-2}K_{-2,C_2} = 0$. 

1.34
The proof of the Gap Theorem (continued)

Now we have to deal with $m = -2$.

$C^{-2,C_2}$ is

$$
\begin{array}{ccc}
-2 & -3 & -4 \\
Z & \xrightarrow{\Delta} & Z[C_2] & \xrightarrow{1-\gamma} & Z[C_2]
\end{array}
$$

Passing to fixed points gives

$$
\begin{array}{ccc}
-2 & -3 & -4 \\
Z & \xrightarrow{1} & Z & \xrightarrow{0} & Z
\end{array}
$$

This has nontrivial homology, but only in dimension $-4$, so again $\pi_{-2} K_{-2,C_2}^C = 0$.

This completes the proof of the Gap Theorem.
The proof of the Gap Theorem (continued)

Now we have to deal with $m = -2$.

$C^{-2,C_2}$ is

\[
\begin{array}{ccc}
-2 & -3 & -4 \\
\mathbb{Z} & \xrightarrow{\Delta} & \mathbb{Z}[C_2] & \xrightarrow{1-\gamma} & \mathbb{Z}[C_2] \\
\end{array}
\]

Passing to fixed points gives

\[
\begin{array}{ccc}
-2 & -3 & -4 \\
\mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \\
\end{array}
\]

This has nontrivial homology, but only in dimension $-4$, so again $\pi_{-2}K^{C_2}_{-2,C_2} = 0$.

This completes the proof of the Gap Theorem. $2 + 2 = 4$