Mike Hill
Mike Hopkins
Doug Ravenel

Inside the proof of the Kervaire invariant theorem
or
How I got bitten by the equivariant bug

Math 549

May 1, 2015

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Mike Hopkins
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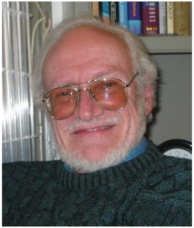

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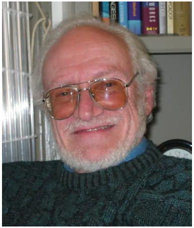

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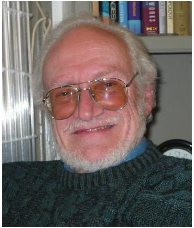

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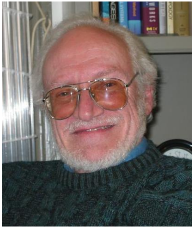

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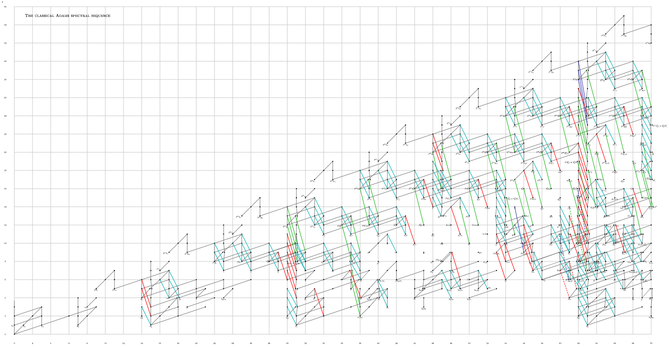

Chart by Dan Isaksen

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Prelude (continued)

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1931-2013

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-  In the 70s, Nishida extended these ideas to show that each positive dimensional element of $\pi_* S^0$ is nilpotent.
-  In the 80s, Devinatz, Hopkins and Smith leveraged these ideas still further to prove the Nilpotence Theorem in stable homotopy theory.

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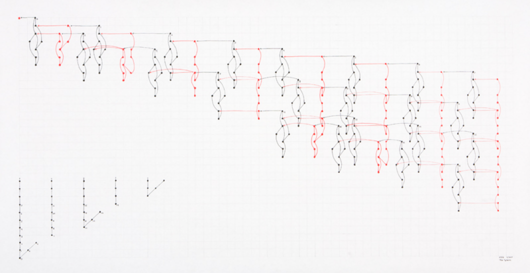

Drawing by Bob Bruner

The HHR strategy

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If $\theta_7 \in \pi_{254}(S^0)$ exists, (i) implies it has a nontrivial image in this group, so it cannot exist. The argument for θ_j for larger j is similar, since $|\theta_j| = 2^{j+1} - 2 \equiv -2 \bmod 256$ for $j \geq 7$.

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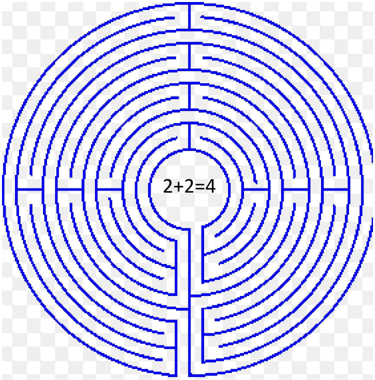

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$$LY = G_+ \underset{H}{\wedge} Y \qquad \text{and} \qquad RY = \prod_{|G/H|} Y,$$

where G permutes the H-invariant factors Y. It is useful to consider a similar functor using the smash product, namely

$$N_H^G Y := \bigwedge_{|G/H|} Y,$$

the norm functor on Y.

Representation spheres

Let V be a finite dimensional orthogonal representation of G.

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where $\mathrm{Ind}_H^G W$ denotes the induced representation $\mathbf{R}[G] \otimes_{\mathbf{R}[H]} W$.

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We denote the resulting C_2-spectrum by $MU_{\mathbf{R}}$.

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Real cobordism (continued)

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Constructing our spectrum Ω

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Calculations show that there is an element $\Delta \in \pi_{256}^G D^{-1}MU^{((G))}$ such that the induced map

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The slice spectral sequence

How do we make such calculations?

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We will call these spectra slice spheres.

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Let $\mathcal{S}^G$ denote the category of G-spectra. Define $P_n\mathcal{S}^G$ to be the subcategory generated by the elements of T_n^G, i.e., by slice spheres of dimension $\geq n$.

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The slice spectral sequence for $G = C_2$ (continued)

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The slice spectral sequence for general groups G

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The slice spectral sequence for general groups G (continued)

We use the resulting filtration of $\mathcal{S}^G$ to define “connective covers” P_nX, “Postnikov sections” P^nX and slices P_n^nX as before.

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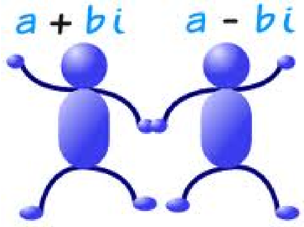

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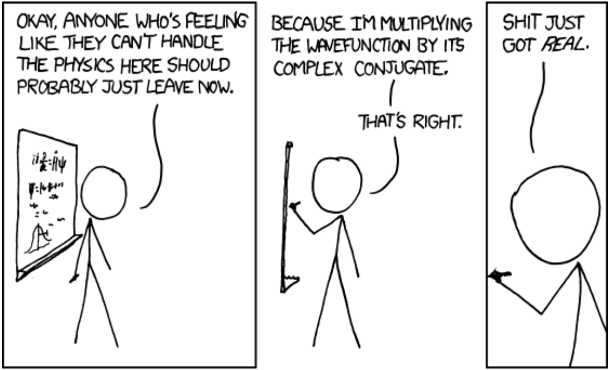

The proof of the Gap Theorem

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$\pi_{-2}\Omega$ never had a chance!

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The proof of the Gap Theorem (continued)

How do we compute $\pi_* K^G_{m,H}$?

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The proof of the Gap Theorem (continued)

It follows that

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We now analyze $C^{m,H}$ and $(C^{m,H})^G$ for $m \geq 0$.

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The proof of the Gap Theorem (continued)

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$$(G_+ \underset{H}{\wedge} S^{m\rho_H})^G = \begin{cases} S^m & \text{for } H = G \\ * & \text{otherwise.} \end{cases}$$

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The proof of the Gap Theorem (continued)

The bottom G-cell of $G_+ \underset{H}{\wedge} S^{m\rho_H}$ is

$$(G_+ \underset{H}{\wedge} S^{m\rho_H})^H = G_+ \underset{H}{\wedge} S^m$$

in dimension m, while the top cell is in dimension $m|H|$. Similar statements hold for $C^{m,H}$, $C^{-m,H}$ and their fixed point subcomplexes.

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For the Gap Theorem we want to show that $\pi_{-2} K^G_{m,H} = 0$ in all cases. From the above we see that the only values of m we need to consider are $m = -1$ and $m = -2$.

The proof of the Gap Theorem (continued)

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The proof of the Gap Theorem (continued)

For the Gap Theorem we want to show that $\pi_{-2} K^G_{m,H} = 0$ in all cases, and the only values of m we need to consider are $m = -1$ and $m = -2$.

For simplicity I will do this for $H = G = C_2$, this being similar in essence to the cases where $G = C_8$.

For $m = 1$, C^{1,C_2} is the reduced C_2-cellular chain complex for S^{ρ_2}. It is

$$\begin{array}{ccc} 1 & & 2 \\ \mathbf{Z} & \xleftarrow{\ \nabla\ } & \mathbf{Z}[C_2] \end{array}$$

where ∇ is the augmentation map sending the generator γ to 1.

The proof of the Gap Theorem (continued)

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Its $\mathbf{Z}$-linear dual C^{-1,C_2} is

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The proof of the Gap Theorem (continued)

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This has trivial homology, so $\pi_{-2}K^{C_2}_{-1,C_2} = 0$.

The proof of the Gap Theorem (continued)

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This has nontrivial homology, but only in dimension -4, so again $\pi_{-2} K_{-2,C_2}^{C_2} = 0$.

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This has nontrivial homology, but only in dimension -4, so again $\pi_{-2}K^{C_2}_{-2,C_2} = 0$.

This completes the proof of the Gap Theorem. $2 + 2 = 4$