Definition: A spectrum \( E \) is a sequence of spaces \( E_n \) with maps

\[
\sum E_n \xrightarrow{f_n} E_{n+1}.
\]

or equivalently maps

\[
E_n \xrightarrow{g_n} \Sigma E_{n+1}.
\]

In a suspension spectrum, each \( f_n \) is an equivalence. In an \( \Omega \)-spectrum, each \( g_n \) is an equivalence.

For a pointed space \( X \) there is a suspension spectrum where the \( n \)th space is the \( n \)th suspension of \( X \).

Any spectrum can be converted to an \( \Omega \)-spectrum by replacing \( E_n \) with the space

\[
E_n = \lim_{k \to \infty} \Sigma^k E_{n+k}.
\]

The sphere spectrum \( S^0 \) is the suspension spectrum for \( X=S^0 \).

The original example of an \( \Omega \)-spectrum is the Eilenberg-Mac Lane spectrum for an abelian group \( G \) in which \( E_n=K(G, n) \).

The reduced suspension of a pointed space \( X \) is the double cone on \( X \) with the line through the base point \( x_0 \) collapsed to a point, which is the new base point.

A map \( f: \Sigma X \to Y \) is equivalent to a map \( g:X \to \Omega Y \). For each \( x \) in \( X \), \( f \) determines a closed path in \( Y \). Hence we get a map \( g \) as claimed since \( \Omega Y \) is the space of closed paths in \( Y \).
In a spectrum we have maps \( g_n: E_n \rightarrow \Omega E_{n+1} \)

\[
\begin{align*}
E_n & \xrightarrow{g_n} SE_{n+1} \\
& \xrightarrow{2} SE_{n+2} \\
& \xrightarrow{2} SE_{n+3} \rightarrow \ldots
\end{align*}
\]

Call the limit \( E \). It follows that \( E_n \cong SE_{n+1} \).

A map between spectra \( E \rightarrow F \) could be a sequence of maps \( E_n \rightarrow F_n \) compatible with the structure maps. This is too restrictive!

EXAMPLE. Consider the suspension spectra \( S^1 \) and \( S^0 \). In the former, the \( n \)th space is \( S^{n+1} \), and the latter it is \( S^n \). We have the Hopf map of spaces \( \eta: S^3 \rightarrow S^2 \). However \( \eta \) is NOT the suspension of any map \( S^2 \rightarrow S^1 \).

Classifying spaces

Let \( G \) be a topological group. Let \( E_n G \) be the \((n+1)\)th fold join of \( G \) with itself. Given spaces \( X \) and \( Y \), their join \( X * Y \) is the quotient of the product \( X \times I \times Y \), where \( I = [0,1] \), where

\[
(x', 0, y) \sim (x'', 0, y) \quad \text{for any } x', x'' \in X, y \in Y;
\]

and

\[
(x', 1, y) \sim (x', 1, y') \quad \text{if } x \in X, y' \in Y.
\]

Exercises: \( S^m * S^n = S^{m+n+1} \). For \( G = \mathbb{C}_2 \), \( E_n G = S^n \) with antipodal action. In general \( E_n G \) is \((n-1)\)-connected.

Definition \( G \) acts freely on \( E_n G \) by left multiplication \( n \) each coordinate in \( G \). Let \( B_n G \)
be its orbit space. Eg B_1 G unreduced suspension of G. Let the classifying space BG be the limit of the B_n G and EG the limit of the E_n G. EG is contractible and has a free G action. This construction is functorial in G, i.e. a homomorphism G --> H induced a map BG --> BH.

When G is discrete and acts freely on a space X, we get an equivariant map X \to EF and a map from X/G to BG. The homotopy class of the latter determines the group action.

Consider the cases G=O(n) or U(n), the nth orthogonal or unitary group. We have spaces BO(n) and BU(n). They come equipped with n-dimensional real or complex vector bundles \gamma_n and \gamma^C_n. Each has an associated unit disk bundle D and unit sphere bundle S. Consider the space D/S. This is called a Thom space. Call them MO(n) and MU(n). We can use them to construct two spectra MO and MU.

In the orthogonal case, MO_n = MO(n). We need a map \sum MO(n) --> MO(n+1). Consider the direct sum gamma'_n of the vector bundle \gamma_n with a trivial line bundle. It is an (n+1)-plane bundle induced by the map BO(n)-->BO(n+1) incuced by the inclusion of O(n) into O(n+1). The Thom space for \gamma'_n is \sum MO(n), so we have our map from it to MO(n+1).

In the unitary case, a similar construction gives a map \sum^2 MU(n) --> MU(n+1). We define the spectrum MU by MU_{2n}=MU(n) and MU_{2n+1}=\sum MU(n).

The spectra MO and MU are very important.

Remark: In each example spectrum E so far, the space E_n is (n-1)-connected. This definition does NOT require this. Such a spectrum is said to be connective.

Example of a nonconnective spectrum:

Bott Periodicity Theorem: \Omega^2 BU is equivalent to Z x BU. It implies that \pi_k BU =\pi_{k+2} BU. The orthogonal analog is \Omega^8 BO is equivalent to Z x BO, so \pi_k BO =\pi_{k+8} BO.

We can use this to construct spectra K and KO. K_{2n}=Z x BU, K_{2n+1}=U. For any G, \Omega BG = G, so \Omega K_{2n+2}=K_{2n+1}, and \Omega^2 K_{2n+2}=K_{2n}. This leads to an \Omega-spectrum K. It is NOT connective.

To continued Monday 3:25 in 1101. Read Intro and Chapter 1 of Lewis-May-Steinberger.