SERRE SPECTRAL SEQUENCE EXERCISES

MATH 549
SPRING 2015

These problems should be done by groups of up to three people. Each team should pick a different problem. Then one member of the team should present the solution to the class as a short talk. You may be able to find some clues online and you are free to ask me for help.

Be prepared to present your solutions the week of February 16.

1. Let $U(n)$ denote the n-dimensional unitary group.
   (a) Evaluating a matrix on a fixed unit vector leads to a map $U(n) \to S^{2n-1}$ with fiber $U(n-1)$. Use this fiber sequence to compute $H^*(U(n); \mathbf{Z})$ by induction on n.
   (b) Now consider the fiber sequence
   $$U(n) \simeq \Omega BU(n) \to PBU(n) \simeq * \to BU(n)$$
   where PX and ΩX denote the path and loop spaces for the pointed space X. Use the Serre spectral sequence to compute $H^*(BU(n); \mathbf{Z})$.
2. Let λ denote the canonical complex line bundle over $\mathbf{C}P^\infty$. Recall that
   $$H^*(\mathbf{C}P^\infty; \mathbf{Z}) = \mathbf{Z}[x] \qquad \text{where } x \in H^2$$
   and that the Euler class (also known as the first Chern class) of λ is $e(\lambda) = c_1(\lambda) = x$. Now consider the n-fold tensor product $\lambda^{\otimes n}$. It is known that $e(\lambda^{\otimes n}) = nx$. It is also know that the total space of the corresponding S^1-bundle is $K(\mathbf{Z}/n, 1)$.
   (a) Let p be an odd prime. Use the fiber sequence above to compute
   $$H^*(K(\mathbf{Z}/p, 1); \mathbf{Z}/p).$$
   (b) Now consider the fiber sequence
   $$K(\mathbf{Z}/p, 1) \simeq \Omega K(\mathbf{Z}/p, 2) \to PK(\mathbf{Z}/p, 2) \simeq * \to K(\mathbf{Z}/p, 2).$$
   Use the Serre spectral sequence to find $H^*(K(\mathbf{Z}/p, 2); \mathbf{Z}/p)$. I did this for $p = 2$ in class on 2/2/15.
3. Consider the fiber sequence
   $$K(\mathbf{Z}, 2) = \mathbf{C}P^\infty \longrightarrow X \longrightarrow S^3 \longrightarrow K(\mathbf{Z}, 3)$$
   where the map on the right corresponds to the generator of $H^3(S^3; \mathbf{Z})$. The space X is the 3-connected cover of S^3. From the long exact sequence of homotopy groups we see that
   $$\pi_i X = \begin{cases} 0 & \text{for } i \leq 3 \\ \pi_i S^3 & \text{otherwise.} \end{cases}$$
   We also know that $H^3(X; \mathbf{Z}) = 0$. Use the Serre spectral sequence to compute $H^*(X; \mathbf{Z})$.