The Serre spectral sequence

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Suppose we have a fiber sequence

$F \xrightarrow{i} E \xrightarrow{p} B$ $\quad F = p^{-1}(b) \quad b \in B$

where p is a FIBRATION (to be defined shortly) and F is its FIBER. Suppose we know $H^*(B)$ and $H^*(F)$ and we want to find the H^*E.

$F \longrightarrow E \longrightarrow B$

$$\begin{tikzcd} H^*F \arrow[dr] & & \\ & \boxed{\text{Serre spectral sequence}} \arrow[r] & H^*E. \\ H^*B \arrow[ur] & & \end{tikzcd}$$

Example $S^1 \xrightarrow{i} S^3 \xrightarrow{p} S^2$ $\quad S^2 = \mathbb{C}P^1$ $\quad$ FIBER BUNDLE

p is the HOPF FIBRATION

What is a FIBRATION?

HUREWICZ's definition. A map $p: E \to B$ is a FIBRATION if for any diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "\beta"] \arrow[ur, dashed, "h"] & B \end{tikzcd} \qquad \begin{matrix} x \\ \downarrow \\ (x,0) \end{matrix}$$

$\exists\, h$ with $hi = \alpha$ and $ph = \beta$

h is called a LIFTING of β.

SERRE's definition: We need this only for $X = I^n$ for all n.

Related definition: E is a FIBER BUNDLE over B with FIBER F if each $b \in B$ has a neighborhood U s.t. $p^{-1}(U) \cong U \times F$ so that the map $p|p^{-1}(U)$ is projection onto first factor.

What is the SERRE SPECTRAL SEQUENCE?

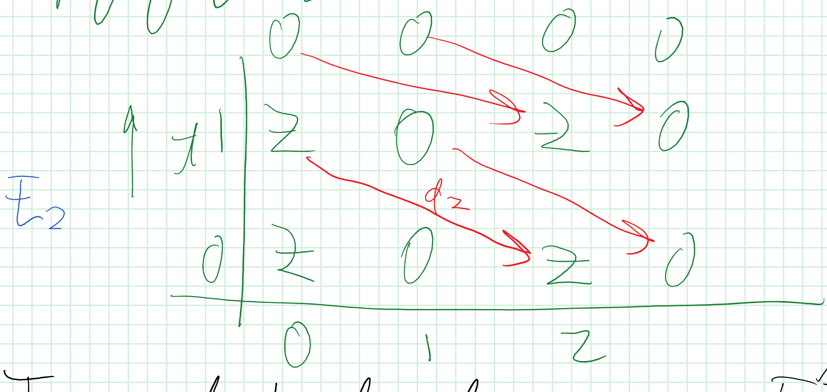
$$E_2^{s,t} = H^s(B; H^t F) \rightsquigarrow H^*E$$

a BIGRADED GROUP

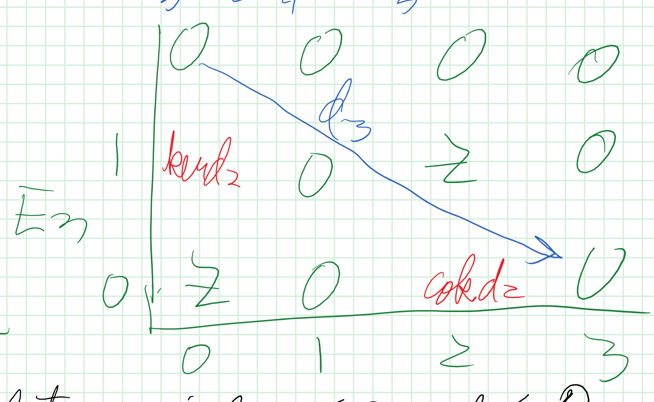
We will define more such bigraded groups $E_r^{s,t}$ for $r = 2, 3, 4, 5, \ldots$ and homomorphisms DIFFERENTIALS

$$d_r : E_r^{s,t} \longrightarrow E_r^{s+r,\, t-r+1}$$

Hopf fibration



$E_3 = E_4 = E_5 = \cdots$



Formal properties: $E_r^{s,t} = 0$ if $s < 0$ or $t < 0$

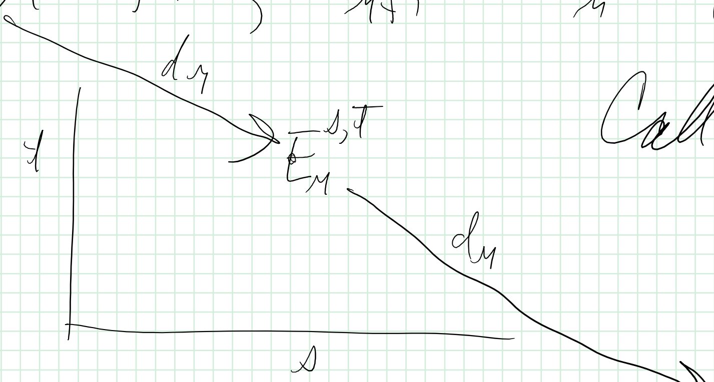
① $d_r d_r = 0$ $\qquad E_r^{s-r,\,t+r-1} \xrightarrow{d_r^{s-r,\,t+r-1}} E_r^{s,t} \xrightarrow{d_r^{s,t}} E_r^{s+r,\,t+1-r}$

② $E^{s,t} := \ker d^{s,t} / \operatorname{im} d^{s-r,\,t+r-1}$

② $E_{r+1}^{s,t} := \ker d_r^{s,t} / \operatorname{im} d_r^{s-r,t+r-1}$

If both $d_r^{s,t}$ and $d_r^{s-r,t+r-1}$ is trivial, then $E_{r+1}^{s,t} = E_r^{s,t}$

③ For each s,t, $E_{r+1}^{s,t} = E_r^{s,t}$ for $r >> 0$.



Call this gp $E_\infty^{s,t}$

④ $E_\infty^{s,t}$ is a subquotient of $H^{s+t}(E)$

One might think that

$$H^n E = \bigoplus_{0 \le s \le n} E_\infty^{s,n-s} \quad \text{(NOT TRUE)}$$

(true for field coefficients)

What is true (LATER)

In our example

$$H^i(E) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ \ker d_2 & i = 1 \\ \operatorname{coker} d_2 & i = 2 \\ \mathbb{Z} & \text{for } i = 3 \\ 0 & \text{for } i > 3 \end{cases}$$

$H^i(S^3)$

Our d_2 is an isomorphism, so $\ker d_2 = 0 = \operatorname{coker} d_2$

New example. $S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n$

$n=2$

$$\begin{tikzcd}
1 & \mathbb{Z} \arrow[drr, "d_2'"] & 0 & \mathbb{Z} \arrow[drr, "d_2''"] & 0 & \mathbb{Z} \\
0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

$E_2^{s,t} = H^s(\mathbb{C}P^2; H^t S^1)$ Each d_2 is an isomorphism

More formal properties

(5) Naturality

$$\begin{tikzcd}
S^1 \arrow[r] & S^5 \arrow[r] & \mathbb{C}P^2 \\
S^1 \arrow[r] \arrow[u, equal] & S^3 \arrow[r] \arrow[u] & \mathbb{C}P^1 \arrow[u]
\end{tikzcd}$$

This induces a map of SSS's.

$\rightsquigarrow$ d_2' is an isomorphism

(6) Differentials play nicely with products (to be continued)