Tuesday, January 23, 2018 1:53 PM

Cup products + the Serre SS

Suppose we have a fibration $F \xrightarrow{i} E \xrightarrow{p} B$

assume B is simply connected.

The Serre SS converging to H^*E has

$$E_2^{p,q} = H^p(B; H^q F)$$

Recall $H^* X$ is a ring under cup product.

This means that $E_2^{*,*}$ is also a ring under cup product.

Assume that either

1) We are using cohomology with field coefficients ($k = \mathbb{Q}$ or $\mathbb{Z}/p$ for p a prime)
2) We are using integer coefficients and both H^*F and H^*B are free abelian gps

In either case $H^p = (B; H^q F) = H^p(B) \otimes H^q(F)$

and $E_2 \cong H^*(B) \otimes H^*(F)$ as algebras.

<u>Thm</u> a) d_2 in the Serre SS is a DERIVATION

$$d_2(xy) = d_2(x)y + (-1)^{|x|} x d_2(y)$$

$|x|$ means dimension of x

b) (E_2, d_2) is a bigraded cochain complex, and its cohomology E_3 inherits a multiplication and d_3 is a derivation.

c) Similarly each d_r is a derivation.

EILENBERG-MACLANE Spaces

Thm Let A be a discrete abelian gp and $n \geq 0$. There is a space called $K(A,n)$ with
$$\pi_i\, K(A,n) = \begin{cases} A & \text{for } i = n \\ 0 & \text{for } i \neq n \end{cases}$$
It is unique up to homotopy equivalence.

Thm $H^n(X;A) \cong$ set of homotopy classes of maps $X \longrightarrow K(A,n)$.

This set has a natural abelian gp structure induced by a map $K(A,n) \times K(A,n) \longrightarrow K(A,n)$ making it a topological abelian gp.

Examples
1) $K(\mathbb{Z},1) = S^1$
2) $K(\mathbb{Z}/2,1) = \mathbb{R}P^\infty$
3) $K(\mathbb{Z},2) = \mathbb{C}P^\infty$

Recall there is a fiber sequence
$$S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n$$

$$\begin{array}{ccccc} \| & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & S^{2n+3} & \longrightarrow & \mathbb{C}P^{n+1} \end{array}$$

Let $n \to \infty$ and take colimit

$$S^1 \longrightarrow S^\infty \longrightarrow \mathbb{C}P^\infty$$

$\pi_i S^\infty = 0$ for all $i > 0$. S^∞ is contractible

<u>Thm</u> For any fiber sequence $F \xrightarrow{i} E \xrightarrow{p} B$ there is a long exact sequence

$$\cdots \longrightarrow \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \longrightarrow \pi_{n-1} F \longrightarrow \cdots$$

<u>Cor</u> If $\pi_n E = 0$ for all n, then $\pi_n B \cong \pi_{n-1} F$ for all $n > 0$.

<u>The path fibration</u> Let (X, x_0) be a path conn space with base point. Let PX, the path space of X, be the space of paths in X starting at x_0, i.e. of maps $(I, 0) \longrightarrow (X, x_0)$

Prop PX is contractible, i.e. the map

$$PX \longrightarrow pt \longrightarrow PX$$

$* \longmapsto$ constant path at x_0

Proof The map above $PX \to PX$ is homotopic to identity on PX, i.e. there is a homotopy

$$PX \times I \xrightarrow{h} PX \ni w$$

s.t. $h(w, 0)$ = constant path in X

$h(w, 1) = w$

$$h(w,s)(t) = w(s,t) \qquad 0 \le s, t \le 1$$

QED

Prop The map $PX \xrightarrow{e} X$ given by $e(w) = w(1)$ is a fibration. The preimage of x_0, the space of paths that start and end at x_0 in ΩX, the LOOP SPACE of X. Then

$$\Omega X \longrightarrow PX \longrightarrow X$$

is a fiber sequence. Hence

$\pi_n(\Omega X) \cong \pi_{n+1} X$.

Cor If $X = K(A, n+1)$ then $\Omega X = K(A, n)$

We want to study the Serre SS

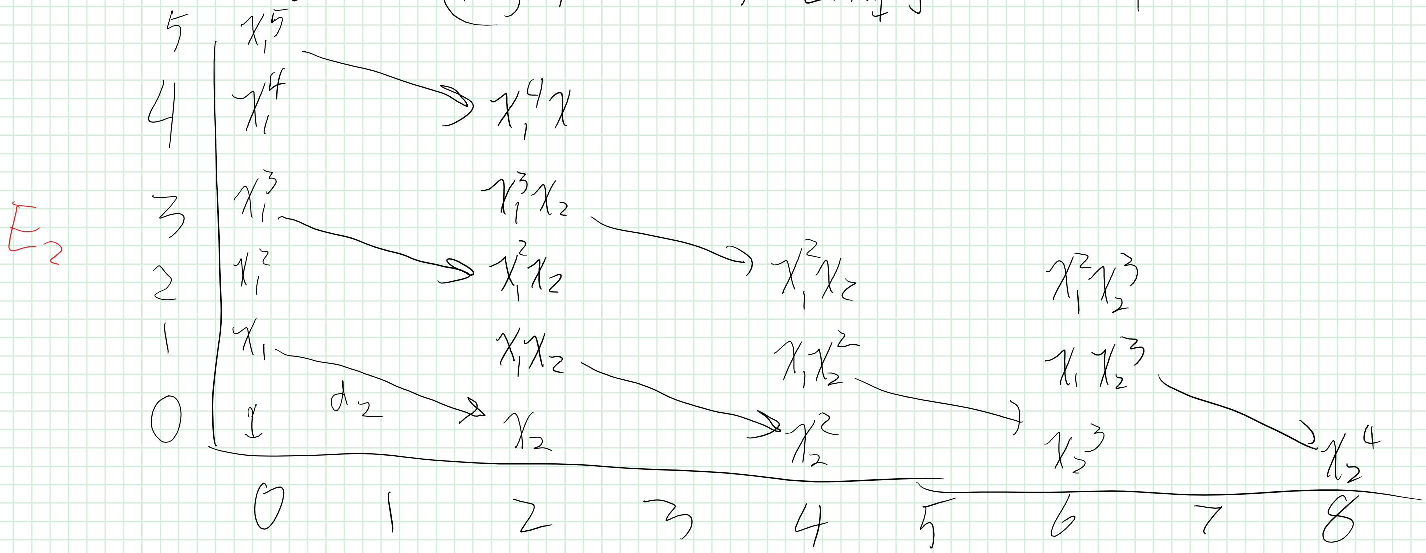
$$F = K(\mathbb{Z}/2, 1) \longrightarrow \overset{E}{\underset{\text{path space}}{X}} \longrightarrow K(\mathbb{Z}/2, 2) = B$$

$F = \mathbb{RP}^\infty$

$\mathbb{R}P$ space

We will use it to determine $H^*(B;\mathbb{Z}/2)$

We know $H^*(F;\mathbb{Z}/2) = \mathbb{Z}/2[x_1]$ $\quad x_1 \in H^1$



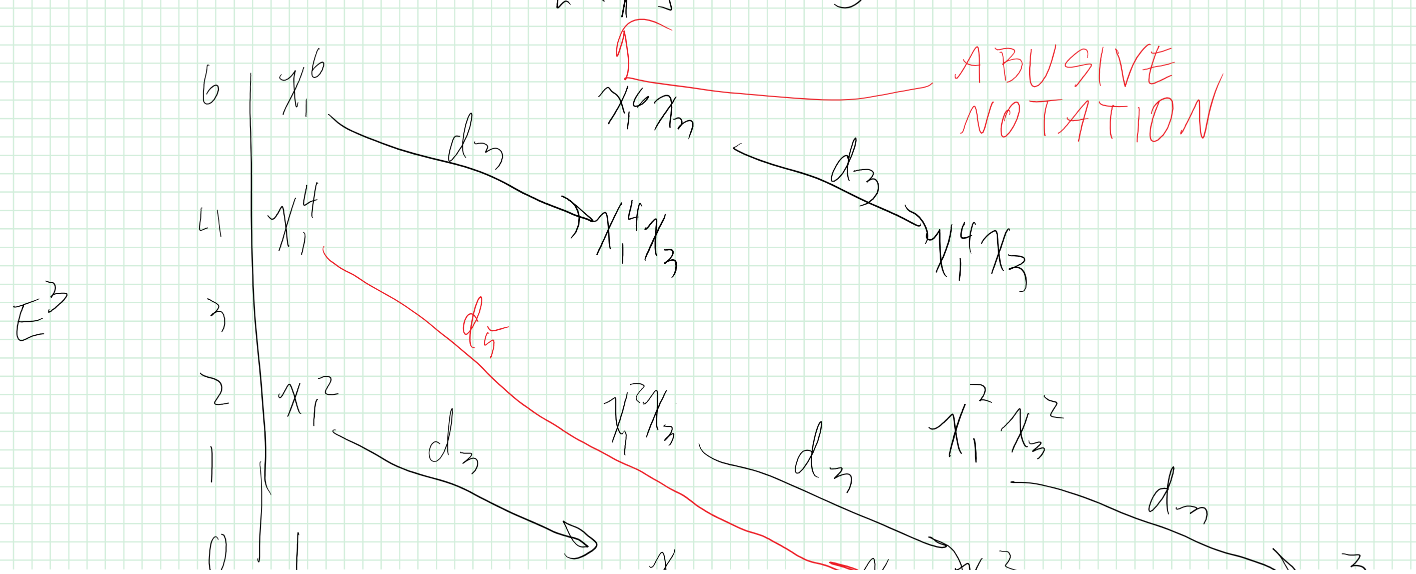
$\exists\, x_2 \in H^2 B$ with $d_2(x_1) = x_2$

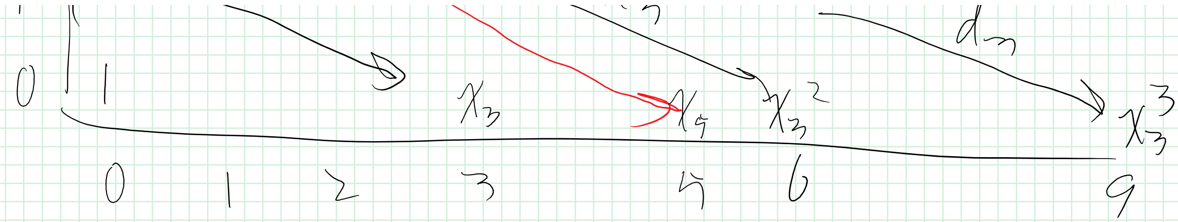
$$d_2(x_1^n) = n\, x_1^{n-1} x_2 = \begin{cases} 0 & \text{if } n \text{ is even} \\ x_1^{n-1} x_2 & \text{if } n \text{ is odd} \end{cases}$$

All powers of x_2 must be $\neq 0$ and

$$d_2(x_1^i x_2^j) = \begin{cases} 0 & \text{if } i \text{ even} \\ x_1^{i-1} x_2^{j+1} & \text{if is is odd} \end{cases}$$

This leaves $\mathbb{Z}/2[x_1^2] \subset E_3$





We get $x_3 \in H^3 B$ with $d_3(x_1^2) = x_3$

We conclude that $d_3\left(x_1^{2i} x_3^j\right) = \begin{cases} 0 & i \text{ even} \\ x_1^{2i-2} x_3^{j+1} & i \text{ odd} \end{cases}$

This leave $\mathbb{Z}/2[x_1^4] \subset E^4$, so $\exists\, x_5 \in H^5 B$, etc.

Conclusion: $x_i \in H^i$

$$H^*(B;\mathbb{Z}/2) = \mathbb{Z}/2[x_2, x_3, x_5, x_9, \dots x_{1+2^j} \dots]$$
$$\| $$
$$H^*(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$$

See Ch 9 of Mosher + Tangora and 3rd paper of Serre