Adams method for computing $\pi_{n+k} S^n$ for $n >> 0$.

Let $X_0 = S^n$ and $L_0 = K(\mathbb{Z}_2, n)$ and $f_0 : S^0 \to L_0$ is the map inducing isomorphism in π_n. Let X_1 denote the fiber of f_0.

Hence we know $\pi_{n+i} X_1 = \begin{cases} \pi_{n+i} S^n & \text{for } i > 0 \\ 0 & \text{for } i \leq 0. \end{cases}$

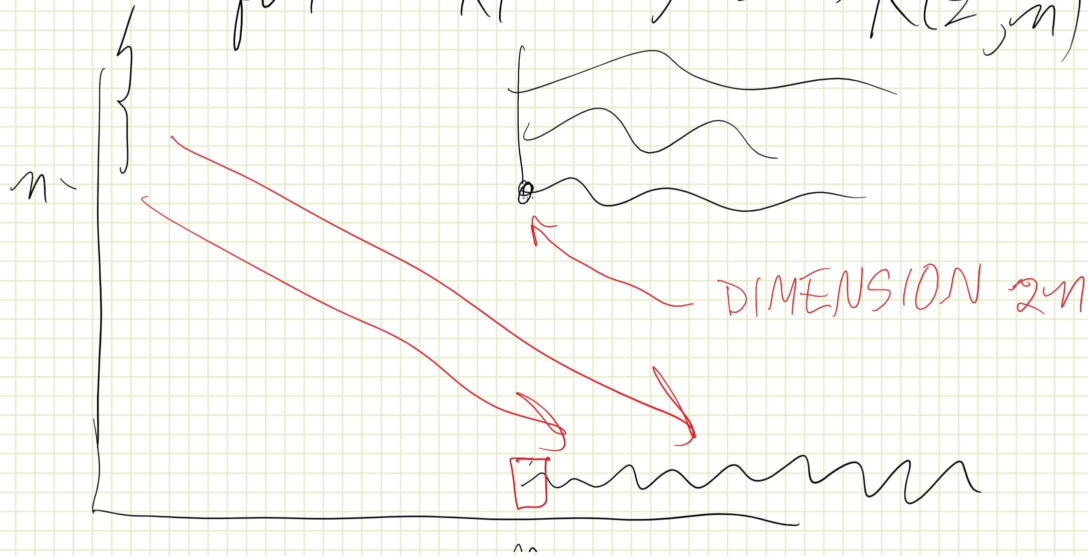
and the Serre SS for $X_1 \to X_0 \to L_0$ leads to a short exact sequence in $H^*(-;\mathbb{Z}/2)$

DIMENSION SHIFT

$$0 \leftarrow H^* S^n \leftarrow H^* L_0 \leftarrow H^* X_1 \leftarrow 0$$

$$0 \leftarrow H^{n+i} S^n \leftarrow H^{n+i} L_0 \leftarrow H^{n+i-1} X_1 \leftarrow 0$$

The Serre SS for $X_1 \to S^n \to K(\mathbb{Z}_2, n)$



Serre's approach: Note $\pi_n X_n = \pi_{n+1} S^n \cong \mathbb{Z}/2$, so there is a map $X_1 \longrightarrow K(\mathbb{Z}/2, n)$ and we can compute the H^* of its fiber, and so on.

mod 2 Steenrod algebra

Adams approach: Find a set of A-module generators $H^*(X_1; \mathbb{Z}/2)$. This leads to a map $X_1 \xrightarrow{f_1} L_1 =$ product of $K(\mathbb{Z}/2, n+i)$s

The product has one factor for each A-module generator of $H^* X_1$

Hence $H^*(f_1)$ is onto. Let X_2 be the fiber of f_1, so we get another short exact sequence

$$0 \leftarrow H^{n+i} X_1 \longleftarrow H^{n+i} L_1 \longleftarrow H^{n+i-1} X_2 \longleftarrow 0$$

KNOWN H^* and π_*

Repeat this process with X_2, i.e. find a set of A-module generators of H^*X_2 and use them to construct a map $X_2 \xrightarrow{f_2} L_2$

We get a diagram

$$\begin{tikzcd} S^n = X_0 \arrow[d, "f_0"] & X_1 \arrow[l] \arrow[d, "f_1"] & X_2 \arrow[l] \arrow[d, "f_2"] & X_3 \arrow[l] \arrow[d, "f_3"] & \cdots \arrow[l] \\ K(\mathbb{Z}_2, n) = L_0 & L_1 & L_2 & L_3 & \end{tikzcd}$$

ADAMS RESOLUTION

(1) X_{s+1} is the fiber of f_s

(2) $H^*(f_s)$ is onto

(3) $H^*(L_s)$ for $s>0$ is a free A-module.

For each $s \geq 0$ we have a fiber sequence

$$X_{s+1} \longrightarrow X_s \xrightarrow{f_s} L_s$$

Its Serre SS leads to a SES (for $i \leq n$)

(1) $$0 \leftarrow H^{n+i} X_s \leftarrow H^{n+i} L_s \leftarrow H^{n+i-1} X_{s+1} \leftarrow 0$$

<u>Theorem</u> $\pi_{n+i}(S^n)$ is a finite abelian group for $0 < i < n-1$, independent of n.

We are getting information about the 2-components of these groups

Algebraic exercise: Given 2 short exact sequences of abelian gps

$$0 \leftarrow A_0 \xleftarrow{\alpha_0} B_0 \xleftarrow{\beta_0} A_1 \leftarrow 0$$

and $0 \leftarrow A_1 \xleftarrow{\alpha_1} B_1 \xleftarrow{\beta_1} A_2 \leftarrow 0$, the following is also exact

$$0 \leftarrow A_0 \xleftarrow{\alpha_0} B_0 \xleftarrow{\beta_0 \alpha_1} B_1 \xleftarrow{\beta_1} A_2 \leftarrow 0,$$

the SPLICE of the two short exact sequences above. Given a third SES

$$0 \leftarrow A_2 \xleftarrow{\alpha_2} B_2 \xleftarrow{\beta_2} A_3 \leftarrow 0$$

we get

$$0 \leftarrow A_0 \xleftarrow{\alpha_0} B_0 \xleftarrow{\beta_0 \alpha_1} B_1 \xleftarrow{\beta_1 \alpha_2} B_2 \xleftarrow{\beta_2} A_3 \leftarrow 0$$

We can splice the sequences of (1) for $s > 0$ and get a long exact sequence

$$0 \leftarrow H^{n+i} X_1 \leftarrow H^{n+i} L_1 \leftarrow H^{n+i-1} L_2 \leftarrow H^{n+i-2} L_3 \leftarrow \cdots$$

(2) $$0 \leftarrow H^* X_1 \leftarrow H^* L_1 \xleftarrow{0} H^* L_2 \xleftarrow{0} H^* L_3 \xleftarrow{0} \cdots$$

free A-modules

an A-module that is not free

$H^* K(\mathbb{Z}/2, n+i)$ = free A-module on a class in dimension $n+i$

$$\pi_* K(\mathbb{Z}/2, n+i) = \begin{cases} \mathbb{Z}/2 & \text{for } * = n+i \\ 0 & \text{for other } * \end{cases}$$

Note $\pi_* K(\mathbb{Z}/2, n+i) = \mathrm{Hom}_A(H^* K(\mathbb{Z}/2, n+i), \mathbb{Z}/2)$

$\pi_* X$ is COVARIANT in X

$H^* X$ is CONTRAVARIANT in X

$\mathrm{Hom}_A(H^* X ; M)$ is covariant in X.

Hence $\pi_* L_s \cong \mathrm{Hom}_A(H^* L_s, \mathbb{Z}/2)$

Algebraic digression: Let R be a ring (e.g. A) and M an R-module.

We can choose a free R-resolution of M

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots$$

where each F_i is a free R-module and the sequence is exact.

Let N be another R-module.

Apply the functor $\mathrm{Hom}_R(-, N)$ to the sequence

$$F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow F_3 \leftarrow \cdots$$

We get a cochain complex

$$\mathrm{Hom}_R(F_0, N) \to \mathrm{Hom}_R(F_1, N) \to \mathrm{Hom}_R(F_2, N) \to \cdots$$

Theorem The cohomology of this cochain depends only on M and N, and NOT on the choice of free resolution. $H^i(\ \ ) = \mathrm{Ext}^i_R(M, N)$

FACT For $R = \mathbb{Z}$ (or any principal ideal domain), $\mathrm{Ext}^i(-,-) = 0$ for $i > 1$.
If R is a field, then $\mathrm{Ext}^i = 0$ for $i > 0$.
For other rings, such as $R = A$ we could have $\mathrm{Ext}^i \neq 0$ for all $i \geq 0$.

END OF DIGRESSION

Consider the case $R = A$, $M = H^* X$, and $N = \mathbb{Z}/2$

We get $\mathrm{Ext}_R(M, N) = \mathrm{Ext}_A(H^* X, \mathbb{Z}/2)$

VERY COMPLICATED TO COMPUTE

This Ext is the E_2-term of the

1958 Adams spectral sequence for finding the 2-component of $\pi_* X_1 = \pi_* S^n$ for $n >> 0$.