Tuesday, February 6, 2018 1:50 PM

Recall the mod 2 Steenrod algebra A.
It has basis $\{Sq^I : I \text{ admissible}\}$ where
$I = (i_1, i_2, \cdots i_\ell)$ $Sq^I = Sq^{i_1} Sq^{i_2} \cdots Sq^{i_\ell}$
ℓ = length of I
ADMISSIBILITY means $i_k \geq 2 i_{k+1}$ for $1 \leq k < \ell$.
The EXCESS $e(I) = i_1 - i_2 - i_3 \cdots - i_\ell \geq 0$

$$= (i_1 - 2i_2) + (i_2 - 2i_3) \cdots + (i_{\ell-1} - 2i_\ell) + i_\ell$$

For I empty, $e(I) = 0$

Prop For $x \in H^n(-)$ then

$$Sq^I(x) = \begin{cases} 0 & \text{if } e(I) > n \\ (x)^n & \text{if } e(I) = n \\ ? & \text{if } e(I) < n \end{cases}$$

See Ch. 9 of Mosher + Tangora

Thm Let $K_n = K(\mathbb{Z}/2, n)$ and let $x_n \in H^n K_n$ be the fundamental class (see below)
Then $H^*(K_n; \mathbb{Z}/2) = \mathbb{Z}/2[Sq^I x_n : e(I) < n]$

What is the fundamental class?
Recall $[X, K_n]$ =: set of hty classes of maps $X \to Y$
has a natural group structure and

is iso to $H^n(X;\mathbb{Z}/2)$.

Let $X = K_n$. $H^n(K_n;\mathbb{Z}/2) \approx [K_n, K_n]$

$$\chi_n \longleftrightarrow 1_{K_n}$$

For $n=1$, the Theorem $H^*K_1 = \mathbb{Z}/2[\chi_1]$, as we knew before

For $n=2$ we need to find all I with $e(I)=1$. They are $\{(1), (2,1), (4,2,1) \dots\dots\}$

$H^*K_2 = \mathbb{Z}/2[\chi_2, Sq^1\chi_2, Sq^2Sq^1\chi_2, \dots\dots]$

Set of I with $e(I)=2$

$(2), (4,2), (8,4,2) \dots\dots$

$(3,1), (6,3,1), (12,6,3,1) \dots\dots$

$(5,2,1), (10,5,2,1), (20,10,5,2,1) \dots\dots$

$(9,4,2,1) \dots\dots$

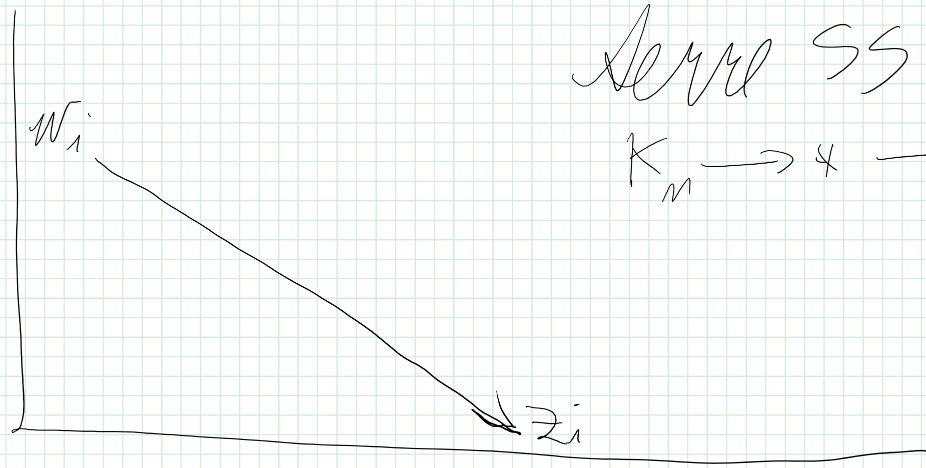
$(17,8,4,2,1) \dots\dots$

Theorem says $H^*(K_3) = \mathbb{Z}/2[Sq^I\chi_3 : e(I)<3]$

Sketch of proof. We use the Borel Theorem. It says if $\{w_i\}$ is a simple system

of generators for $H^* K_n$, then
$H^* K_{n+1} = \mathbb{Z}/2 [z_i]$ where $|z_i| = 1+|w_i|$
and z_i is the TRANSGRESSION of w_i

Serre SS for
$K_n \longrightarrow * \longrightarrow K_{n+1}$



Will argue by induction on n. We know it is true for $n=1$.
Suppose $H^* K_n$ is as claimed.
A simple system of generators (SSG) for a polynomial algebra $\mathbb{Z}/2[a_1, a_2, \dots]$ is $\{ a_i^{2^j} : i > 0, j \geq 0 \}$
We need to do this to $H^* K_n$.

<u>example</u> $n=1$ $\quad H^* K_1 = \mathbb{Z}/2[x_1]$
a SSG is $\{ x_1, x_1^2, x_1^4, x_1^8, \dots \}$

$= \{x_1, Sq^1 x_1, Sq^2 Sq^1 x_1, Sq^4 Sq^2 Sq^1 x_1, \ldots \}$

$= \{ Sq^I x_1 : e(I) \le 1 \}$

Similarly this process converts

$H^* K_n = \mathbb{Z}/2 [Sq^I x_n : e(I) < n]$

to the SSG $= \{ Sq^I x_n : e(I) \le n \}$

Borel theorem says

$H^* K_{n+1} = \mathbb{Z}/2 [Sq^I x_{n+1} : e(I) \le n]$

Let $L_n = K(\mathbb{Z}, n)$. Want to describe

$H^*(L_n; \mathbb{Z}/2)$. We know $K(\mathbb{Z},1) = S^1$ and

$K(\mathbb{Z},2) = \mathbb{C}P^\infty$.

SSG

$H^*(L_1) = E(x_1) = \mathbb{Z}/2[x_1]/(x_1^2)$ $\{x_1\}$

$H^*(L_2) = \mathbb{Z}/2[x_2]$ $\{x_2, x_2^2, x_2^4, \ldots\}$

$= \{x_2, Sq^2 x_2, Sq^4 Sq^2 x_2, \ldots\}$

Note $Sq^1 x_2 = 0$ in $H^* L_2 = H^* \mathbb{C}P^\infty$.

Hence $Sq^I x_2 = 0$ for any I with $i_\ell = 1$.

<u>Thm</u> $H^*(L_n;\mathbb{Z}/2) = \mathbb{Z}/2\left[Sq^I x_n : e(I) < n \text{ and } i_\ell \neq 1\right]$

Can be proved by induction on n as before.

An alternate approach. There is a fiber sequence $K_{n-1} \to L_n \xrightarrow{[2]} L_n \to K_n$

Recall there is a natural iso

$$[X, L_n] \cong H^n(X;\mathbb{Z})$$

e.g. $[L_n, L_n] \cong H^n(L_n;\mathbb{Z}) = \mathbb{Z}$

$$1_{L_n} \longleftrightarrow \text{generator}$$

$$[2] \longleftrightarrow 2 \text{ generator}$$

We have a map $K_n \xrightarrow{b} L_{n+1}$

If $y_{n+1} \in H^{n+1}(L_{n+1};\mathbb{Z}/2)$ is the mod 2 fundamental class

then $b^* y_{n+1} = Sq^1 x_n \in H^{n+1}(K_n;\mathbb{Z}/2)$

The map $b^* : H^* L_{n+1} \to H^*_n K_n$ is 1-1

Its image in $H^*K_n = \mathbb{Z}/2[Sq^I x_n : e(I) \le n]$

$$Sq^J y_{n+1} \longmapsto Sq^J Sq^1 x_n$$

where J is admissible with $j_1 > 1$

$$e(Sq^J Sq^1) = e(J) - 1$$

so $e(J) = 1 + e(Sq^J Sq^1)$

Since $e(Sq^J Sq^1) < n$, $e(J) \le n$.

This leads to another proof.

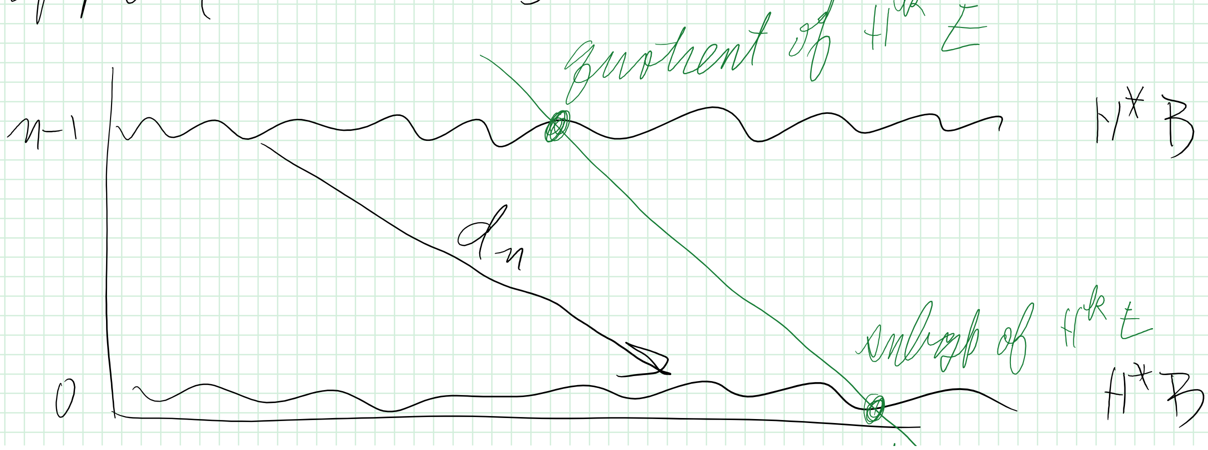
Three easy cases of the Serre SS

Recall that for a fiber seq. $F \to E \to B$

we a spectral sequence with

$$E_2^{p,q} = H^p(B; H^q F) \Rightarrow H^{p+q}(E)$$

① Suppose $F = S^{n-1}$



$E_{n+1} = E_\infty$. For each k there is a SES

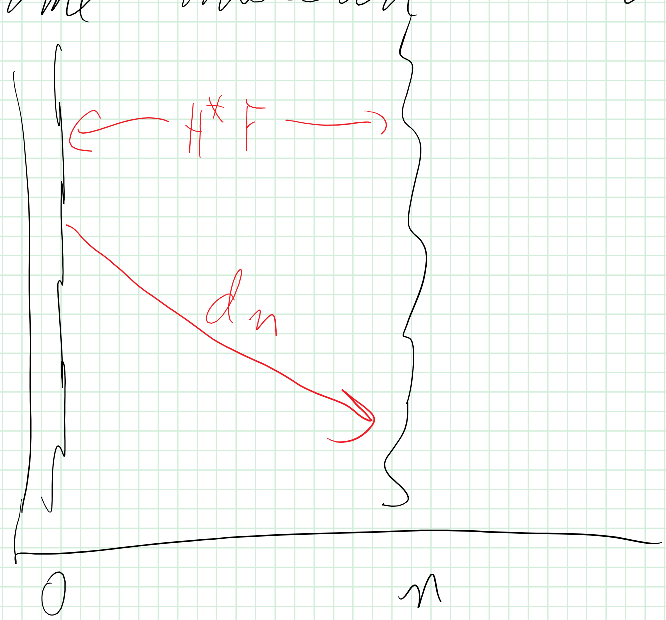
$$\begin{tikzcd} 0 \arrow[r] & \operatorname{coker} d_n \arrow[r] & H^kE \arrow[r] & \ker d_n \arrow[r] \arrow[d, phantom, "\subset"] & 0 \\ & H^kB \arrow[u] & & H^{k-n}B & \end{tikzcd}$$

There is also a LES

$$H^{k-1}E \longrightarrow H^iB \xrightarrow{d_n} H^{i+n}B \longrightarrow H^kE$$

GYSIN SEQUENCE

(2) Assume instead that $B = S^n$

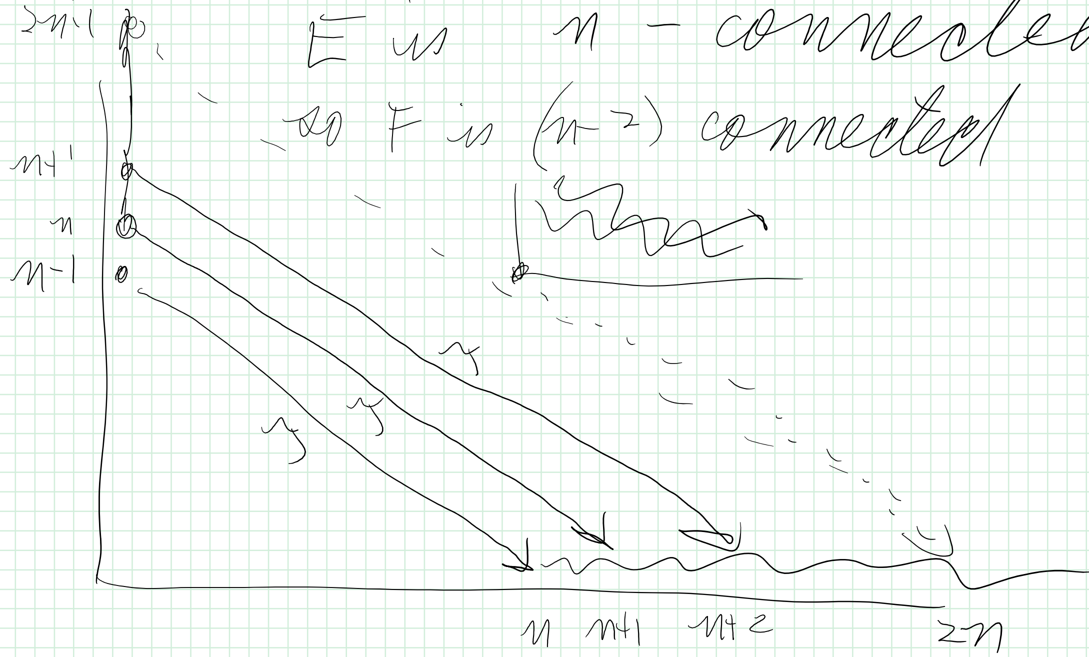


We get a LES

$$H^kE \longrightarrow H^kF \xrightarrow{d_n} H^{k+1-n}F \longrightarrow H^{k+1}E$$

WANG SEQUENCE

③ Suppose B is $(n-1)$-connected and E is n-connected so F is $(n-2)$ connected



Below dim $2n-1$ we have a LES

$$\to H^i E \to H^i F \xrightarrow{\tau} H^{i+1} B \longrightarrow H^{i+1} E \to$$

Familiar example $\quad K_{n-1} \longrightarrow \underset{E}{\overset{*}{\|}} \longrightarrow \underset{B}{K_n \atop \|}$

We get

$$0 \longrightarrow H^i K_{n-1} \xrightarrow{\cong} H^{i+1} K_n \longrightarrow 0$$

for $i+1 < 2n-1$