The EILENBERG-MOORE SS

Suppose we have a fiber sequence

(1)
$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "p"] & B \\ F \arrow[r] & E_f \arrow[u] \arrow[r, "p'"] & X \arrow[u, "f"'] \end{tikzcd}$$

Question: Find H^*E_f in terms of the other 3 spaces in the square

E_f = pullback of p and f

$= \{(e,x) \in E \times X : p(e) = f(x)\}$

$p'(x) \simeq F$ for any $x \in X$

Serre SS designed to compute H^*E in terms of H^*B and H^*F.

Theorem (Eilenberg-Moore 1966). There a SS converging to H^*E_f with

$E_2^{-p,q} = \mathrm{Tor}^{p,q}_{H^*B}(H^*X, H^*E)$ with

$E_r^{-p,q} \xrightarrow{d_r} E_r^{r-p,\,q+1-r}$

Both H^*X and H^*E are modules over H^*B.

Cor Let $X = *$ in (1) and $H^*(-) = H^*(-;k)$ k a field. Then

$E_2 = \mathrm{Tor}_{H^*B}(k, H^*E)$

If the top row of (1) is the path fibration for E, then we get a SS converging to $H^*\Omega B$ with

$E_2 = \mathrm{Tor}_{H^*B}(k, k)$.

How to compute this Tor group.

Suppose $R = H^*B = H^*S^n = E(x_n)$ = exterior algebra

We need a projective R-resolution of k

$$\begin{array}{cccccccccccc} & & 0 & & 1 & & 2 & & 3 & \\ 0 \leftarrow k \leftarrow & & R & \xleftarrow{x_n} & \Sigma^n R & \xleftarrow{x_n} & \Sigma^{2n} R & \xleftarrow{x_n} & \Sigma^{3n} R & \leftarrow \cdots \end{array}$$

$0 \leftarrow 1 \; x_n$

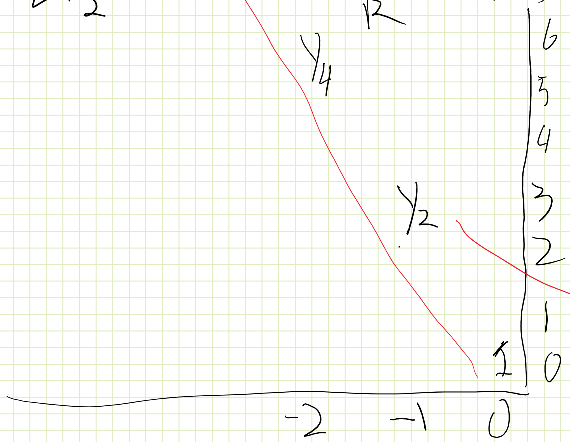
To find Tor, apply $- \otimes_R k$ to the chain cx and take its homology

$$\begin{array}{ccccccccc} 0 & & 1 & & 2 & & 3 & \\ k & \xleftarrow{0} & \Sigma^n k & \xleftarrow{0} & \Sigma^{2n} k & \xleftarrow{0} & \Sigma^{3n} k & \leftarrow \cdots \end{array}$$

$$\mathrm{Tor}_R^{p,q}(k,k) = \begin{cases} k & \text{if } q = np \\ 0 & \text{else} \end{cases}$$

Let $n = 3$. In the SS we have

$$E_2^{-p,q} = \mathrm{Tor}_R^{p,q}(k,k) = \begin{cases} k & \text{if } q = np \\ 0 & \text{else} \end{cases}$$



$y_2 \in E_2^{-1,3}$.

all differentials vanish.

Conclusion $H^i(\Omega S^n; k) = \begin{cases} k & \text{if } i = p(n-1) \text{ for some } p \geq 0 \\ 0 & \text{else.} \end{cases}$

$\}0$ else.

Mysterious fact. $\mathrm{Tor}_R(k,k)$ has a multiplicative structure, i.e. there is a map

$$\mathrm{Tor}_R^{p',q'}(k,k)\otimes \mathrm{Tor}_R^{p'',q''}(k,k) \to \mathrm{Tor}_R^{p'+p'',q'+q''}(k,k).$$

In the case at hand, let

$$Y_{p(n-1)} \in \mathrm{Tor}_R^{p,pn}(k,k)$$

Then $Y_{p'(n-1)} \otimes Y_{p''(n-1)} \longrightarrow \binom{p'+p''}{p'} Y_{(p'+p'')(n-1)}$

We find $H^* \Omega S^n = \Gamma(Y_{n-1})$, as we saw with the Serre SS.

More about how to compute $\mathrm{Tor}_R(k,k)$

graded tensor product

$|x|=n$

bigraded tensor product

(2) Suppose $R = k[x]$. Then we have a projective resolution

$$0 \leftarrow k \xleftarrow{\varepsilon} R \xleftarrow{x} \Sigma^{n} R \leftarrow 0$$
$$0 \longleftarrow\!\shortmid\, x$$

$$\rightsquigarrow \quad \mathrm{Tor}_{k[x]}(k,k) = E(\bar{x})$$

$$\bar{x} \in \mathrm{Tor}^{1,n}_{R}(k,k)$$

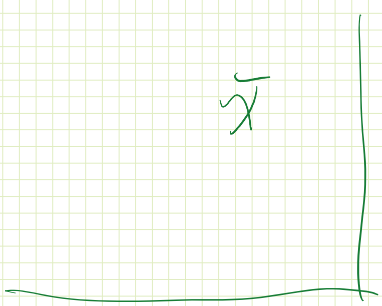
Example

$$\begin{tikzcd} K(\mathbb{Z},1) \arrow[r] & * \arrow[r] & K(\mathbb{Z},2) = \mathbb{C}P^{\infty} \\ S^1 \arrow[u, equal] \arrow[r, equal] & S^1 \arrow[u] \arrow[r] & * \arrow[u] \end{tikzcd}$$

$R = H^{*}\mathbb{C}P^{\infty} = \mathbb{Z}[x]$

$x \in H^2$

$$\mathrm{Tor}^{p,q}_{R}(\mathbb{Z},\mathbb{Z}) = \begin{cases} \mathbb{Z} & p = q = 0 \\ \mathbb{Z} & p = 1 \quad q = 2 \\ 0 & \text{else} \end{cases}$$



$\bar{x} \in E_2^{-1,2}$

$$H^{i}(S^1;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0,1 \\ 0 & \text{else} \end{cases}$$

Will compute $H^{*}(\Omega^2 S^3; \mathbb{Z}/2)$

$B = \Omega S^3$

$$H^*(B;\mathbb{Z}/2) = \Gamma(x_2)$$

$$= E(x_2, x_4, x_8, x_{16}, \dots)$$

$$= E(x_2)\otimes E(x_4)\otimes E(x_8)\otimes \cdots$$

$$\otimes\ L \qquad = \underset{i\geq 1}{} E(x_{2^i})$$

This means

$$\mathrm{Tor}_{H^*\Omega S^3}(\mathbb{Z}/2,\mathbb{Z}/2) = \bigotimes_{i\geq 1} \mathrm{Tor}_{E(x_{2^i})}(\mathbb{Z}/2,\mathbb{Z}/2)$$

$$\bigotimes_{i\geq 0} \Gamma(x_{2^i-1})$$

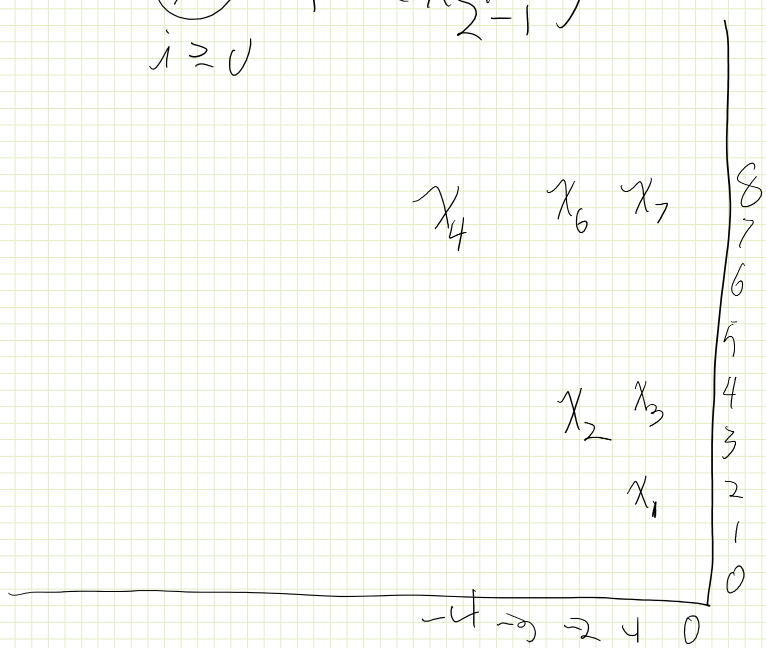
$x_{2^i-1} \in \mathrm{Tor}^{1,2^i}$

$x_2 \in \mathrm{Tor}^{2,4} = E_2^{-2,4}$

$x_4 \in E^{-4,8}$

$x_3 \in E^{-1,4}$

$d_3\ E_3^{-4,8} \longrightarrow E_3^{-1,6}$



Recall $\Gamma(x) \cong E(x, \gamma_2(x), \gamma_4(x), \dots)$

where $\gamma_k(x)$ = k th divided power of x

$E \;=\; \Gamma(x \quad\quad E^{-2^j,\,2^{i+j}}$

$2 \qquad \cdot\, i \geq 1, j = 0\,)$

We have some additional structure that differentials must respect. $H^*(\Omega X)$ has a coproduct. In $\Gamma(x)$ the coproduct is given by

$$\gamma_k(x) \xrightarrow{\Delta} \sum_{0 \le i \le k} \gamma_i(x) \otimes \gamma_{k-i}(x)$$

Note $\mathrm{Hom}(\Gamma(x), k) = \mathbb{Z}/2[x^*]$, which implies the above.

Def An element y in a coalgebra (such as $H^*\Omega B$) is PRIMITIVE if its coproduct is $y \otimes 1 + 1 \otimes y$.

Differentials have to respect the coalgebra structure, i.e.

If $\Delta(x) = \sum x_i' \otimes x_i'' = x \otimes 1 + 1 \otimes x +$ other terms

4 $\quad \Delta d_n(x) = \sum d_n(x_i') \otimes x_i'' + x_i' \otimes d_n(x_i'')$

In the SS for $\Omega^2 S^3$, suppose x is the first element where $d_n(x) \neq 0$

Then (4) says

$$\Delta d_r(x) = d_r(x)\otimes 1 + 1\otimes d_r(x)$$

i.e. $d_r(x)$ is primitive.

In our case the only primitives are $x_1, x_3, x_7, \ldots x_{2^i-1}$, so $d_r(x)$ must be one of them. This means that source x with $d_r(x)\neq 0$ must have dimension 2^i-2 for some $i\geq 1$.

Hence x must be

$$x_{2^i-2}\in E_2^{-2,2^i} \quad \text{and} \quad r\geq 2$$

This means there are no nontrivial differentials $E_2=E_\infty$ so

$$H^*(\Omega^2 S^3)=\Gamma(x_1, x_3, x_7, \ldots)$$

so $H_*(\Omega^2 S^3) = \mathbb{Z}/2[x_1^*, x_3^*, x_7^*, \ldots]$

STUDENT TALKS on $\geq 3/27$.

Semin: Milnor's paper on Steenrod algebra for $p=2$

Dionel : "Yoneda's 1954 paper
see also Hilton + Stambach
Kenny : McCleary §7.1 on differential homological algebra.
Keping : McCleary 7.2

Recall $H^*(\Omega S^3;\mathbb{Z}) = \Gamma(x_2)$

$$= \mathbb{Z}\{x_{2i} : i \geq 0\}$$

with $x_{2i} \cdot x_{2j} = \binom{i+j}{i} x_{2(i+j)}$

LUCAS LEMMA (19th century)

Let $a = \sum_{i\geq 0} a_i p^i$ and $b = \sum_{i \geq 0} b_i p^i$

where $0 \leq a_i, b_i < p$

Then $\binom{a}{b} \equiv \prod_{i\geq 0} \binom{a_i}{b_i}$ mod p.

and $\binom{a_i}{b_i} = \begin{cases} \neq 0 & \text{if } 0 \leq b_i \leq a_i \\ 0 & \text{if } b_i > a_i \end{cases}$

Let $p=3$. Then $x_2^2 = 2x_4 \equiv -x_4$

$$x_2^3 = 6x_0 \equiv 0 \mod 3$$

$$x_6 x_2 = \binom{4}{1} x_8 \equiv x_8$$

$$x_{\cancel{6}} x_4 = \binom{5}{2} x_{10} \equiv x_{10}$$

$$\Gamma(x_2) = \mathbb{Z}/p[x_2, x_{2p}, x_{2p^2} \cdots]/(x_{2p^i}^p)$$

$$= \bigotimes_{i \geq 0} \underbrace{\mathbb{Z}/p[x_{2p^i}]/(x_{2p^i}^p)}$$

$T(x) :=$ TRUNCATED POLYNOMIAL ALGEBRA OF HEIGHT p.

Let $R = \mathbb{Z}/p[x]/(x^p)$ with $\dim x = 2n$

Need to find $\mathrm{Tor}_R(\mathbb{Z}/p, \mathbb{Z}/p)$. There is a projective resolution of the form

$$0 \leftarrow \mathbb{Z}/p \xleftarrow{\varepsilon} R \xleftarrow{x} \Sigma^{2n} R \xleftarrow{x^{p-1}} \Sigma^{2pn} R \xleftarrow{x} \Sigma^{(2p+2)n} R \cdots$$

$$0 \leftarrow\!\shortmid x \qquad 0 \leftarrow\!\shortmid x$$

$$0 \leftarrow\!\shortmid x^{p-1} \qquad 0 \leftarrow\!\shortmid x^{p-1}$$

$$\mathrm{Tor}_R^{i,j}(\mathbb{Z}/p, \mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p & \text{if } i = 2k \text{ and } j = 2pkn \\ \mathbb{Z}/p & \text{if } i = 2k+1 \text{ and } j = 2(kp+1)n \\ 0 & \text{else} \end{cases}$$

For $n = 1$ we get gps in dimensions

$0, 2, 6, 8, 12, 14, 18, 20, 24, 26, \cdots$

$$\mathrm{Tor}_R^{**}(\mathbb{Z}/p, \mathbb{Z}/p) = E(y_{2n-1}) \otimes \Gamma(y_{2pn-2})$$

Want to use EMSS to find

$H^*(\Omega^2 S^3; \mathbb{Z}/p)$ and $H_*(\Omega^2 S^3; \mathbb{Z}/p)$

$$\mathrm{Tor}_{H^*\Omega^2 S^3}(\mathbb{Z}/p, \mathbb{Z}/p) = \mathrm{Tor}_{\Gamma(x_2)}(\mathbb{Z}/p, \mathbb{Z}/p)$$

$$= \bigotimes_{i \geq 0} \mathrm{Tor}_{T(x_{2p^i})}(\mathbb{Z}/p, \mathbb{Z}/p)$$

$$= \bigotimes_{i \geq 0} E(y_{2p^i-1}) \otimes \Gamma(y_{2p^{i+1}-2})$$

$y_{2p^i-1} \in E_2^{-1, 2p^i}$

$y_{2p^{i+1}-2} \in E_2^{-2, 2p^{i+1}}$

Similar arguments show that

$E_2 = E_\infty$

Conclusion

$$H_*(\Omega^2 S^3; \mathbb{Z}/p) = E(x_1, x_{2p-1}, x_{2p^2-1}, \ldots) \otimes \mathbb{Z}/p[x_{2p-2}, x_{2p^2-2}, \ldots]$$

AMAZING FACT due to Milnor
The dual of the mod p Steenrod
is isomorphic as a graded ring
to the above.

It is possible to compute
$H_*(\Omega^k S^{n+k}; \mathbb{Z}_p)$ for $n>0$ and $k \geq 1$
by similar methods. In each case
the EMSS collapses, i.e.
there are no differentials.
All primes p.

Another SS
EMSS in H_*
ROTHENBERG-STEENROD SS
BAR SS.
Suppose G is a topological group.
It has a classifying space BG
with $\Omega BG \simeq G$. BG can...

be constructed as follows. Suppose we have a space EG with a free action of G and EG is contractible. Then its orbit space EG/G has the right homotopy type to be BG. There is a fiber sequence

$$\begin{array}{ccccc} G & \longrightarrow & EG & \longrightarrow & EG/G. \\ & & \| & & \\ & & * & & \end{array}$$

$$G \simeq \Omega(EG/G) = \Omega BG$$

Examples of EG: $G = C_n$ = cyclic gp of order n.

C_n acts freely on S^{2m-1}, the unit sphere in $\mathbb{C}^m$ via multiplication by $e^{2\pi i/n}$. Consider the maps

$$S^{2m-1} \longrightarrow S^{2m+1} \longrightarrow S^{2m+3} \longrightarrow \cdots$$

$$\mathbb{C}^m \longrightarrow \mathbb{C}^{m+1} \longrightarrow \mathbb{C}^{m+2} \longrightarrow \cdots$$

Let $EC_n = \operatorname{colim} S^{2m-1}$

$m \to \infty$

It is a contractible free G-space.

This also works for $G = S^1$

$$\mathbb{C}^{2m+1}/S^1 = \mathbb{C}P^m$$

so $BS^1 = \mathbb{C}P^\infty$.

Let G be your favorite group.

Def For spaces X and Y, the JOIN $X * Y = X \times I \times Y / \sim$

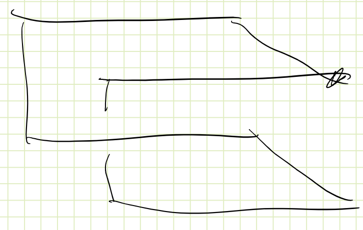
$(x, 0, y) \sim (x', 0, y)$ and

$(x, 1, y) \sim (x, 1, y')$ for all x, x', y, y'

Exercise 1) $S^m * S^n = S^{m+n+1}$

2) If X is $(m-1)$-connected and Y is $(n-1)$-connected then $X * Y$ is $(m+n)$-connected

3) $C_2 * C_2 = S^1$



Let $E_n G = \underbrace{G * G * G \cdots * G}_{n+1 \text{ factors}}$

$= G^{n+1} \times I^n / \sim$

$= G^{n+1} \times \Delta^n / \sim$

It is $(n-1)$-connected and it has a free G-action induced by multiplication of each co-ord in G^{n+1}

We have maps $E_n G \to E_{n+1} G \to E_{n+2} G \to \cdots$

Let $B_n G = E_n G / G$

and $BG = \operatorname{colim} B_n G$

$E_\infty G$ is a a contractible free G-space